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Stochastic and Quantum Space-Time Metrics and the Weak-Field Limit

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In this review we present a simple method of introducing stochastic and quantum metrics into gravitational theory at short distances in terms of small fluctuations around a classical background space-time. We consider only residual effects due to the stochastic (or quantum) theory of gravity and use a perturbative stochastization (or quantization) method. By using the general covariance and correspondence principles, we reconstruct the theory of gravitational, mechanical, electromagnetic, and quantum mechanical processes and tensor algebra in the space-time with stochastic and quantum metrics. Some consequences of the theory are also considered, in particular, it indicates that the value of the fundamental length l lies in the interval $10^{-23} \lesssim l \lesssim 10^{-22}$ cm.

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PREFACE

Recent developments of the experimental technology of high-energy physics permitting one to probe deeper and deeper into matter up to distances of 10^{-16}–10^{-17} cm (even $\sim 10^{-18}$ cm, if energies can be achieved of up to 20 TeV in the center-of-mass system, using the proposed U.S. superconducting supercollider complex) and theoretical work devoted to the construction of a unified field theory of elementary particle forces, including gravitation, have resulted in a deeper understanding of the space-time structure in the microworld. Among the proposed forms of space-time in the microworld [for example, the concept of superspace and pregeometry, lattice (discrete) and cellular structures of space-time, higher-dimensional geometry, etc.], an important role is played by stochastic or quantum space-time. This idea is based on the fact that the quantum fluctuations in geometry are inescapable if one believes in the quantum principle and Einstein's theory.

To realize these possible structures of space-time at short distances we present a working model of stochastic and quantum fluctuations of the space-time metric and consider their consequences. This review deals specifically with a concrete mechanism for space-time metric fluctuations due to background stochastic and gravitonlike quantized fields. Our construction of the theory is based on the general covariance principle, which allows us to obtain physical equations and quantities in space-time with stochastic or quantum fluctuations in the metric. At the same time, our scheme may be useful for taking into account the gravitational force in particle physics phenomena, by means of stochastic and quantum fluctuations in the metric.

It is generally accepted that quantum or stochastic gravitational effects show up essentially at the so-called "Planck mass" of about 10^{19} GeV [the "Planck length" is $l_{\rm Pl} = (\hbar G/c^3)^{1/2} = 10^{-33}$ cm]. From the practical point of view this length is so small that the contribution made by the quantum gravitational effect to any physical quantity is in fact negligible at present attainable energies. However, from our considerations it is clear that between distances of 10^{-33} cm and 10^{-16} cm there may exist some domain in which stochastic and quantum structures of space-time may be manifested. This domain is characterized by the length $l \sim 10^{-23}$–10^{-22} cm.

The purpose of this work is modest. We consider only residual effects due to the stochastic (or quantum) theory of gravity, based on the assumption that, at short distances, the space-time metric fluctuates (or is quantized) and we use the perturbative stochastization (or quantization) method. It seems that the true quantum theory of gravity is not perturbative. It is argued that quantum general relativity may still exist because strong-coupling effects at short distances contradict the assumption that quantum geometry may be understood in terms of small fluctuations around a classical background space-time.

This review consists of seven sections. In accordance with the general covariance and correspondence principles, a theory of gravitational, mechanical, electromagnetic, and quantum mechanical processes and tensor algebra (Sections 1-7) is reconstructed by using the concept of stochastic and quantum fluctuations in the metric.

Physicists concerned with condensed matter may be interested in the discussion of the physical consequences of introducing stochastic and quantum fluctuations in the metric, and of obtaining lower and upper bounds on the value of the fundamental length. The central matters are the general covariance principle, tensor analysis with stochastic and quantum metrics, the T-product definition of geometric quantized objects, stochastic quantization with distributions, and change of the particle mass and of the Newtonian potential due to stochastic (or quantum) fluctuations in the space-time metric.

NOTATION

Greek ($\alpha, \beta, \gamma, \ldots$) and Latin letters ($i, j, k, \ldots$) run over space-time coordinates 0, 1, 2, 3 and over spatial coordinates 1, 2, 3 only.

The Minkowski metric is defined as

$$\eta_{\mu\nu} = 0 \qquad \text{for} \quad \mu \neq \nu$$

$$-\eta_{00} = \eta_{11} = \eta_{22} = \eta_{33} = 1$$

The product of two four-vectors **p** and **q** with components

$$p = (p_0, \mathbf{p}) = (p_0, p_i), \qquad q = (q_0, \mathbf{q}) = (q_0, q_j) \qquad (i, j = 1, 2, 3)$$

is defined as

$$pq = \eta_{\mu\nu} p^\mu q^\nu = p_\nu q^\nu = -p_0 q_0 + (\mathbf{pq}) = -p_0 q_0 + p_1 q_1 + p_2 q_2 + p_3 q_3$$

Summation is carried out by repeating indices, omitting the symbol of summation. Sometimes the Euclidean metric $\delta^{\mu\nu} = \delta_{\mu\nu}$

$$\delta_{\mu\nu} = 0, \qquad \mu \neq \nu$$

$$\delta_{44} = \delta_{11} = \delta_{22} = \delta_{33}$$

is used.

The equivalent notation

$$f(x) = f(x_0, \mathbf{x}) = f(x_0, x_i)$$

$$g(p) = g(p_0, \mathbf{p}) = g(p_0, p_j) \qquad (i, j = 1, 2, 3)$$

will be used for the functions $f(x)$ and $g(p)$ defined in four-dimensional space-time and momentum space, respectively. Moreover, the following notation is clear:

$$\int d^4x\, f(x) = \int dx_0\, d\mathbf{x}\, f(x_0, \mathbf{x}) = \int dx\, f(x)$$

In this review we use the system of units (with some exceptions) in which the light velocity c and the Planck constant h divided by 2π ($\hbar = h/2\pi$) are equal to unity, i.e., $\hbar = c = 1$.

INTRODUCTION

At present, much attention is being paid to the study of space-time properties at short distances. This results from the fact that, first, construction of the unified theory of elementary particle forces including gravitation requires a deeper understanding of space-time structure (high-dimensional, stochastic, and quantum, etc.) at very high energies (or, equivalently, at small distances), and second, advances in high-energy experimental technology allow us to probe a very small space-time region step by step which is defined by a parameter l with the dimension of length. From the experimental data (Ting, 1982; Barber *et al.*, 1979*a*,*b*, 1980; Bailey *et al.*, 1979; Bartel *et al.*, 1980; Berger *et al.*, 1980) on testing locality properties it follows that our usual space-time concept is valid up to distances of $l \lesssim 10^{-16}$ cm (Kinoshita, 1979; Lautrup *et al.*, 1972; Namsrai and Dineykhan, 1983; Namsrai, 1985; also see Bracci *et al.*, 1983, 1987; Li, 1982; Dineykhan and Namsrai, 1986*b*; Kirzhnits, 1967).

On the other hand, it is indisputable that phenomena in the microworld are quantized, i.e., their properties are described by quantum probabilistic laws, while the space-time structure connected with them becomes continuous, at least up to the above-mentioned distances. The structure of space-time and the physical phenomena within it enter inseparably into human cognition, and their interrelations are those of dialectical unity. This unity gives rise to some hope that the quantum and stochastic natures of space-time properties can exist in the microworld and be discovered ultimately. Given this assumption, the following question arises: At what distances do quantum and stochastic structures of space-time start? This has become more pressing in the light of the development of unified ways of describing the fundamental forces in nature. Although the force of gravity is extremely weak with respect to electromagnetic and weak (or electroweak) and strong forces between elementary particles, it is still nonzero, so that as increasing energies probe deeper and deeper into matter, a level eventually should be reached where quantum gravitational effects appear. This is the so-called "Planck mass" of about 10^{19} GeV (the "Planck length" $l_{\rm Pl} = (\hbar G/c^3)^{1/2} = 1.62 \times 10^{-33}$ cm). It is not to be ruled out that between the distances 10^{-33} cm and 10^{-16} cm there may exist some oasis in which the stochastic and quantum structures of space-time may be manifested. As will indeed be shown below, this expected oasis exists, and the value of the fundamental length lies in the interval 10^{-23}–10^{-22} cm, the upper limit for which was obtained in Namsrai (1986*a*, 1988).

The idea of a quantum and stochastic structure of space-time has been discussed by many authors, particularly beginning from the early stages of the development of field theory and the work devoted to the construction of a finite theory of quantized fields free from ultraviolet divergences.

In the theory of quantized and stochastic space-times it is usually assumed that there is no exact conceptual meaning of definite space-time points, i.e., the components of coordinates $\hat{x}^q_\mu$ and x^s_μ in the corresponding spaces do not commute,

$$[\hat{x}^q_\mu, \hat{x}^q_\nu] \neq 0 \qquad \text{for} \quad \mu \neq \nu$$

and are distributed with a *probability measure* $w(b_E^2/l^2)$, where $x^s_\mu = x_\mu + b^E_\mu$ consists of two parts: a regular part $x_\mu = (x_0, \mathbf{x})$ and a stochastic part $b^E_\nu = (ib_4, \mathbf{b})$ (for details, see Namsrai, 1986*b*). The theory of quantized space-time was first discussed by Snyder (1947*a,b*) and Yang (1947), and subsequently developed by Kadyshevsky (1959, 1962, 1980), Gol'fand (1959, 1962), and Tamm (1965) (also see Leznov, 1967; Kirzhnits and Chechin, 1967). For discussion of various theoretical ideas of space-time structures in the microworld, see Blokhintsev (1973), Prugovečki (1984), and Namsrai (1986*b*) (where earlier references can be found).

Among different possible space-time structures [quantum or discrete (Wilson, 1974; Lee, 1983; Friedberg and Lee, 1983; Fradkin and Tseytlin, 1985; Yamamoto, 1985; Banai, 1984, 1985; Fujiwara, 1980), foamlike (Wheeler, 1964; Hawking, 1978, 1983; Strominger, 1984; also see Misner *et al.*, 1973), code (Finkelstein, 1969, 1972, 1974), cellular (Cole, 1972; Kirillov and Kochnev, 1987), and so on] at short distances, the stochastic or fluctuational character of space-time may become the most probable candidate and the natural arena of future physical theory. Indeed, if one believes in the quantum principle and Einstein's theory, then stochastic or fluctuational properties of space-time should inevitably appear in the microworld. A stochastic space-time, which can be used in constructing theories of elementary particles, was first considered by March (1934, 1937), Markov (1940, 1958), and Yukawa (1966). Some attempts were undertaken to construct quantum field theory in a stochastic space-time (Markov, 1959; Komar and Markov, 1959; Takano, 1961, 1967; Ingraham, 1967; Blokhintsev, 1973, 1975, and references therein). Subsequently, this problem was discussed by Roy Choudhury and Roy (1980), Roy (1979, 1986), Cerofolini (1980), Prugovečki (1984), and Asanov *et al.* (1988). Mathematical spaces with a stochastic metric were considered by Frederick (1976) and Sinha and Roy (1987). A formal definition of a linear space with a random metric, mainly in the Euclidean case, was given by Menger (1942, 1949), Sherstnev (1963), Schweizer (1967), and Schweizer and Sklar (1983). Prugovečki's (1984) monograph is devoted to a consistent unification of relativity and quantum theory based on stochastic spaces. The two-point correlation function of metric fluctuations in de Sitter space was calculated by Antoniadis and Mottola (1986).

Stochastic or quantum geometry plays an important role in representing gauge theories by random surfaces and strings (Polyakov, 1981*a,b*; Gomez, 1982) and in the construction of a unified theory of elementary particle interactions based on the theory of strings and superstrings [see, for details, Green *et al.* (1987)]. For the practical realization of the idea of the stochastic and quantum characters of space-time we distinguish two approaches: the first is based on the assumption that quantum and stochastic properties of space-time can manifest themselves at the stage of arithmetization of events (or all reality is subject to an intrinsic stochasticity inherent in the measurement process caused by the stochasticity of space-time), and the second deals with random metrics. Our previous work (Namsrai, 1986*a*,*b*, 1988) belongs to the first approach and is devoted to the study of physical processes by means of quantum and stochastic space-times with coordinates $\hat{x}^q_\nu$ and x^s_ν, which have played the most important role in constructing the nonlocal theory (Efimov, 1977, 1985) of quantized fields and were given by the very nature of stochastic quantum mechanics (Nelson, 1967; Guerra, 1981).

Stochastic and quantum metrics are considered on a much deeper level, where one should take into account the gravitational effects connected with introducing stochastic and quantum space-times into physics. The present paper is devoted to the study of this problem. Here we reconstruct the theory of mechanical, electromagnetic, and gravitational processes from the point of view of stochastic and quantum fluctuations of the space-time metric. By analogy with Einstein's idea of the unification of space and time, which led to the appearance of the parameter $\beta=(1-v^2/c^2)^{-1/2}$ in the theory, in our scheme all physical quantities depend on the fundamental ratio $f(l^2_{\rm Pl}/l^2)$ with some function f defined by a concrete method of introducing stochastic and quantum properties of the space-time metric. Moreover, a deeper connection between the quantum nature of geometry and gravitonlike quantized fields is shown to exist, and upper and lower limits on the value of the fundamental length l are also obtained, that is, $10^{-23}\lesssim l\lesssim 10^{-22}$ cm. This larger value, with respect to Planck's $l_{\rm Pl}$, gives rise to some hope that quantum or stochastic properties of space-time in the microworld may be discovered in the near future.

Our approach may be regarded as a primitive method of quantization (or stochastization) of gravity, based on the quantum (or stochastic) properties of the space-time metric only, and belongs to the standard perturbative scheme where quantum (or stochastic) geometry may be understood in terms of small fluctuations around a classical background space-time. The theory of gravity coupled to matter is not considered here [see Dineykhan *et al.* (1989), where we attempt to construct the Green functions for scalar particle in the fluctuating space-time metric). The effective action for quantum scalars in a background gravitational field is evaluated by Mann *et al.* (1989) in operator regularization (McKeon and Sherry, 1987), using both the weak-field method and the normal coordinate expansion. Attention is currently being paid to the study of nonperturbative methods in quantum gravity, in which the splitting of the metric into a classical background part and a fluctuating quantum part is not made (Rovelli and Smolin, 1988; Kuchar, 1981, and references therein).

Deeper issues in quantum gravity were presented in the proceedings of the second Oxford symposium edited by Isham *et al.* (1981) and of the 11th International conference on general relativity and gravitation edited by MacCallum (1987). For a review of the present status of perturbative and nonperturbative methods in quantum gravity, see Isham (1987).

The outline of the present review is as follows. In Section 1 we introduce some formal linear integral transformations of coordinates which allow us to formulate a general covariance principle for a fictitious "gravitational" field $\varepsilon_{\mu\nu}(x)$, introduced by means of the stochastic metric concept. With this stochastic metric we reconstruct all consequences of the special theory of relativity. Sections 2 and 3 are devoted to the extension of our formalism to the gravitational field and to the investigation of proper tensor analysis leading to changes to Einstein's equation (Section 5) in stochastic space-time. Gravitational effects due to the stochastic metric on physical processes are discussed in Section 4. Reconstruction of relativity theory with quantum fluctuation of the space-time metric is given in Section 6. Here we show that quantum geometry is indeed caused by quantized gravitonlike fields, i.e., a deeper connection exists between them. In Section 7 we discuss some physical consequences of the theory with stochastic and quantum fluctuations of the space-time metric and obtain a lower limit for the fundamental length l.

1. THE SPECIAL THEORY OF RELATIVITY WITH STOCHASTIC METRIC

1.1. Fictitious "Gravitational" Field, the Equivalence Principle, and Modified Space-Time Metric

The main aim of this section is to formulate the physical principles of introducing stochastic fluctuations of the space-time metric. Generally speaking, an idea of a stochastic or quantum fluctuation in the metric is needed in order to understand the unification mechanism of Einstein's theory of relativity with quantum laws, and is caused by a real physical situation when we use the gravitational vacuum concept (or zero-point radiation field) by analogy with the hypothesis of the existence of the stochastic electromagnetic vacuum (Braffort and Tzara, 1954; Braffort *et al.*, 1965; Marshall, 1965; Boyer, 1975*a*,*b*; see Vigier, 1982; Namsrai, 1986*b*, and references therein). It seems that a universal background or radiation field initially arose from processes in the early universe (the Big Bang), acting on physical objects everywhere, and its form of interaction can be described by means of stochastic or quantum fluctuations of the space-time metric. A typical example is the microwave background radiation as a probe of the contemporary structure as well as the history of the universe (Zel'dovich and Sunyaev, 1980). In addition, a Doppler search for a gravitational background radiation with two spacecraft was presented by Bertotti and Iess (1985). All possible zero eigenvalues for quantum fluctuations in the presence of the instanton have been discussed by Inagaki (1977), while quantum cosmological problems connected with the existence of the cosmological constant were discussed by Weinberg (1989).

We assume here that a gravitational vacuum-like background radiation gives rise to some fictitious "gravitational" field which is described by means of stochastic or quantum fluctuations of the space-time metric. Furthermore, it is suggested that, for this fictitious "gravitational" field, the equivalence principle may be reformulated as follows: At every point of space-time in a fictitious gravitational radiation field one can choose the local inertial system of reference such that in a sufficiently small neighborhood of the given point the laws of nature will have the same form as the (pseudo-) Riemannian coordinates. We call the equivalence principle formulated in this way the modified equivalence principle or the equivalence principle of the first level.

Thus, we consider a freely moving particle under the action of a fictitious "gravitational" radiation field. According to the equivalence principle of the first level reformulated above, there exists a system of reference ξ^α in which a particle moves along an almost *rectilinear trajectory* given by the equation

$$\frac{d^2\xi^\alpha}{d\tau^2}=0 \tag{1.1}$$

where $d\tau$ is the *proper time*

$$d\tau^2=-\eta_{\alpha\beta}\,d\xi^\alpha\,d\xi^\beta \tag{1.2}$$

Now, due to the stochastic fluctuation of the space-time metric caused by some initial stochastic gravitational radiation field, the system of reference ξ^α becomes a curvilinear one x^μ with a stochastic metric, and therefore the coordinates ξ^α of the system of reference free from action are functions of x^ν.

In order to introduce stochastic fluctuations in the metric explicitly, we first define the formal *linear integral transformation* of coordinates leading to the passage from the usual system of reference ξ^α with the Minkowski metric $\eta_{\alpha\beta}$ to the quasi-inertial one x^μ with a *stochastic metric* $g_{\mu\nu}(x)$. Let us consider the formal transformation

$$\xi^\alpha \Rightarrow \xi^\mu = x^\mu + \tfrac{1}{2}\int d\rho\, \theta(x^0-\rho)h_0^\mu(\rho) + \cdots + \tfrac{1}{2}\int d\rho\, \theta(x^3-\rho)h_3^\mu(\rho) \tag{1.3}$$

where $h_\beta^\mu(\rho)$ is an arbitrary second-rank tensor and at the same time is a stochastic function of the argument ρ, and $\theta(x)$ is the *Heaviside function*

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Thus, according to the assumption that the coordinates ξ^α of the system of reference are functions of x^ν, equation (1.1) takes the form

$$\frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) = \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2\xi^\alpha}{\partial x^\nu\,\partial x^\mu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

Multiplying this equation by $\partial x^\lambda/\partial\xi^\alpha$ and using the well-known *multiplication rule*

$$\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial x^\lambda}{\partial\xi^\alpha} = \delta_\mu^\lambda \tag{1.4}$$

we get the following equation of motion:

$$\frac{d^2x^\lambda}{d\tau^2} + \gamma_{\mu\nu}^\lambda\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{1.5}$$

where $\gamma_{\mu\nu}^\lambda$ is the *affine connection*-like quantity defined as

$$\gamma_{\mu\nu}^\lambda \equiv \frac{\partial x^\lambda}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\,\partial x^\nu} \tag{1.6a}$$

The connection

$$\gamma_{\mu\lambda}^\sigma = \tfrac{1}{2}g_{(s)}^{\nu\sigma}\left[\frac{\partial g_{\mu\nu}^{(s)}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}^{(s)}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}^{(s)}}{\partial x^\nu}\right] \tag{1.6b}$$

is easily established in our case (for details, see Section 2), where the metric tensor $g_{\mu\nu}^{(s)}$ is defined by formula (1.9).

The *proper time* (1.2) may also be expressed in the system of reference with the stochastic metric by the formula

$$d\tau^2 = -\eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}dx^\mu\frac{\partial\xi^\beta}{\partial x^\nu}dx^\nu \tag{1.7}$$

or

$$d\tau^2 = -g_{\mu\nu}^{(s)}\,dx^\mu\,dx^\nu \tag{1.8}$$

where $g^{(s)}_{\mu\nu}$ is the *stochastic metric* defined as

$$g^{(s)}_{\mu\nu} \equiv \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta} \tag{1.9}$$

Now, making use of (1.3), we calculate the explicit form of (1.9) and verify its identity (1.4). For this purpose, recalling that the derivative of the Heaviside function $\theta(x)$ equals $\delta(x)$, we get immediately from (1.3)

$$\frac{\partial \xi^\alpha}{\partial x^\mu} = \delta^\alpha_\mu + \tfrac{1}{2}\delta^\beta_\mu h^\alpha_\beta(x^\beta) \equiv \delta^\alpha_\mu + \tfrac{1}{2}\varepsilon^\alpha_\mu(x) \tag{1.10}$$

Therefore

$$g^{(s)}_{\mu\nu}(x) = \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta} = [\delta^\alpha_\mu + \tfrac{1}{2}\varepsilon^\alpha_\mu(x)][\delta^\beta_\nu + \tfrac{1}{2}\varepsilon^\beta_\nu(x)]\eta_{\alpha\beta}$$

$$= \eta_{\mu\nu} + \varepsilon_{\mu\nu}(x) + \tfrac{1}{4}\varepsilon^\rho_\mu(x)\varepsilon_{\nu\rho}(x) \tag{1.11}$$

Here we have assumed that $\varepsilon_{\mu\nu}(x) = \varepsilon_{\nu\mu}(x)$, other properties of which will be given below.

Further, by direct calculation, one can show that an *inverse Jacobian of transformation* with respect to (1.10) is

$$\frac{\partial x^\lambda}{\partial \xi^\alpha} = \delta^\lambda_\alpha - \tfrac{1}{2}\varepsilon^\lambda_\alpha(x) + \tfrac{1}{4}\varepsilon^\rho_\alpha(x)\varepsilon^\lambda_\rho(x) - \tfrac{1}{8}\varepsilon^\rho_\alpha(x)\varepsilon^\delta_\rho(x)\varepsilon^\lambda_\delta(x) + \cdots$$

It turns out that when this series is summed up, the result reads

$$\frac{\partial x^\lambda}{\partial \xi^\alpha} = \delta^\lambda_\rho[\delta^\alpha_\rho + \tfrac{1}{2}\varepsilon^\alpha_\rho(x)]^{-1} \tag{1.12}$$

This identity allows us to define an *inverse metric tensor* by the following formula:

$$g^{\nu\sigma}_{(s)} \equiv g^{\sigma\nu}_{(s)} \equiv \eta^{\alpha\beta} \frac{\partial x^\nu}{\partial \xi^\alpha} \frac{\partial x^\sigma}{\partial \xi^\beta} = \eta^{\nu\sigma} - \varepsilon^{\nu\sigma}(x) + \tfrac{3}{4}\varepsilon^{\nu\rho}(x)\varepsilon^\sigma_\rho(x) - \cdots \tag{1.13a}$$

It is easily verified that

$$g^{\nu\sigma}_{(s)} g^{(s)}_{\varkappa\nu} = \delta^\sigma_\varkappa \tag{1.13b}$$

for the stochastic metric.

1.2. The Euclidean Postulate and Properties of Stochastic Tensor $\varepsilon_{\mu\nu}(x)$

Generally speaking, the stochastic properties of the tensor $\varepsilon_{\mu\nu}(x)$ are defined in the Euclidean domain of variables $x^E_\mu = (x_4 = ix_0, \mathbf{x})$. It is well known that in the Minkowski space-time, an invariant measure $dP[\tau^2]$

depending on the variable $\tau^2 = x_0^2 - \mathbf{x}^2$ and at the same time obeying the condition

$$\int dP\,[\tau^2] = 1$$

does not exist. In the Euclidean case, the behavior of the random field of interest $\varepsilon(x^E)$ is described by a *probability distribution* $P[\varepsilon]$, or equivalently by the moments of the *probability distribution* (we assume here that all the *moments* exist),

$$\langle \varepsilon(x_1^E) \cdots \varepsilon(x_n^E) \rangle = \int [d\varepsilon]\, \varepsilon(x_1^E) \cdots \varepsilon(x_n^E) P[\varepsilon]$$

In this equation x_i^E are points in a four-dimensional Euclidean space $\mathbb{R}^4$, and the integration is over the value of ε at each point in $\mathbb{R}^4$. Here, for simplicity, tensor indices for the value ε are omitted.

The most common probability distribution encountered in practice is a *Gaussian distribution*, which has

$$P[\varepsilon] = \frac{1}{N} \exp\left\{ -\tfrac{1}{2} \iint d^4x\, d^4y\, \varepsilon_{\nu\mu}(x^E) D^{-1}_{\mu\nu,\rho\sigma}(x^E - y^E) \varepsilon_{\rho\sigma}(y^E) \right\}$$

where N is a constant chosen so that $P[\varepsilon]$ is normalized to unity and $D^{-1}_{\mu\nu,\rho\sigma}(x^E - y^E)$ is the inverse of the *two-point correlation*

$$\langle \varepsilon_{\mu\nu}(x^E) \varepsilon_{\rho\sigma}(y^E) \rangle = D_{\mu\nu,\rho\sigma}(x^E - y^E) \tag{1.14}$$

Before going to a definition of its momentum representation, we now give the properties of the tensor field $\varepsilon_{\mu\nu}(x)$. Assume that the stochastic additional term to the usual Minkowski metric given by formula (1.11) is a weak tensor field which should be regarded as a gravitonlike field with spin two. Then, $\varepsilon_{\mu\nu}(x)$ satisfies the following conditions:

$$\begin{aligned} \varepsilon_{\mu\nu}(x) &= \varepsilon_{\nu\mu}(x) \\ \partial_\mu \varepsilon_{\mu\nu}(x) &= 0 \\ \operatorname{Tr} \varepsilon_{\mu\nu}(x) &= 0 \end{aligned} \tag{1.15}$$

These conditions are sufficient to construct the *correlation function* by means of the *divisor*

$$d_{\mu\nu}(p) = \eta_{\mu\nu} - p_\mu p_\nu / p^2$$

satisfying the identities

$$p^\mu d_{\mu\nu}(p) = 0, \qquad \eta^{\mu\nu} d_{\mu\nu}(p) = 3$$

and, therefore,

$$\eta^{\mu\nu}\Pi_{\mu\nu,\varkappa\lambda}(p)=0, \qquad \eta^{\mu\varkappa}\eta^{\nu\lambda}\Pi_{\mu\nu,\varkappa\lambda}(p)=10$$
$$\Pi_{\mu\nu,\varkappa\lambda}(p)\Pi^{\varkappa\lambda}_{\rho\sigma}(p)=\Pi_{\mu\nu,\rho\sigma}(p) \tag{1.16}$$

where

$$\Pi_{\mu\nu,\varkappa\lambda}(p)=d_{\mu\varkappa}(p)d_{\nu\lambda}(p)+d_{\mu\lambda}(p)d_{\nu\varkappa}-\tfrac{2}{3}\,d_{\mu\nu}(p)d_{\varkappa\lambda}(p)$$

is the *projecting tensor* for the spin-two field.

Now we turn to the Euclidean formulation again and define the momentum representation for the *covariance* (1.14), the *Fourier transform* of which is

$$D_{\mu\nu,\rho\sigma}(x^E)=\frac{1}{(2\pi)^4}\int d^4q_E\, e^{-iq_E x^E}\,\tilde{D}^{(i)}_{\mu\nu,\rho\sigma}(q_E) \tag{1.17}$$

Here, we distinguish two possibilities (G is the Newtonian constant):

1. $\tilde{D}^{(1)}_{\mu\nu,\rho\sigma}(q_E)=G^2\Pi^{(1)}_{\mu\nu,\rho\sigma}(q_E)\tilde{D}^{(1)}_l(q_E^2)$
2. $\tilde{D}^{(2)}_{\mu\nu,\rho\sigma}(q_E)=G\Pi^{(2)}_{\mu\nu,\rho\sigma}(q_E)\tilde{D}^{(2)}_l(q_E^2)$,

where the corresponding projecting tensors $\Pi^{(i)}_{\mu\nu,\rho\sigma}(q_E)$ $(i=1,2)$ are defined by means of the divisors

$$d^{(1)}_{\mu\nu}(q_E)=q^E_\mu q^E_\nu-q_E^2\delta_{\mu\nu}, \qquad d^{(2)}_{\mu\nu}(q_E)=q^E_\mu q^E_\nu/q_E^2-\delta_{\mu\nu}$$

respectively, and the *distribution functions* $\tilde{D}^{(1)}_l(q_E^2)$ and $\tilde{D}^{(2)}_l(q_E^2)$ satisfy the following conditions:

$$(2\pi)^{-4}\int d^4q_E\,\tilde{D}^{(1)}_l(q_E^2)=1 \tag{1.18a}$$

$$l^2(2\pi)^{-4}\int d^4q_E\,\tilde{D}^{(2)}_l(q_E^2)=1 \tag{1.18b}$$

where l is the parameter of the theory; we call it the *fundamental length.* The appearance of the value l^2 in expression (1.18b) follows from a dimensional argument.

One can accomplish the passage from the Euclidean description to physical vacuumlike amplitudes; for this the following complex substitutions are used:

$$x_4\to ix^0, \qquad q_4\to -iq^0, \qquad \delta_{\mu\nu}\to\eta_{\mu\nu} \tag{1.19}$$

As a result, the function (1.17) acquires the following form in the Minkowski space-time:

$$D_{\mu\nu,\rho\sigma}(x)=i^{-1}(2\pi)^{-4}\int d^4q\, e^{-iqx}\tilde{D}^{(i)}_{\mu\nu,\rho\sigma}(q), \qquad i=1,2 \tag{1.20}$$

where $\tilde{D}^{(i)}_{\mu\nu,\rho\sigma}(q)$ are constructed by means of the divisors $d^{(1)}_{\mu\nu}(q) = q_\nu q_\mu - q^2\eta_{\mu\nu}$ $[d^{(2)}_{\mu\nu}(q) = q_\mu q_\nu/q^2 - \eta_{\mu\nu}]$ and functions $D^{(i)}_l(q^2)$ and $d^4q = dq_0\, d\mathbf{q}$, $q^2 = -q_0^2 + \mathbf{q}^2$. Here, it should be noted that conditions (1.18a) and (1.18b) satisfy only the Euclidean metric.

Now our main question arises: how to link a real physical construction of the theory with stochastic fluctuations in the metric with the Euclidean description (1.17) or its Minkowski version (1.20). We act as follows. First, we mention that physical observables are indeed obtained by means of some formal averaging procedure in the Minkowski space-time which is equivalent to the *covariance* (1.17) for the Euclidean formulation. For example, the real physical meaning of the obtained metric (1.11) is its *averaged version*

$$g_{\mu\nu}(x) = \langle g^{(s)}_{\mu\nu}(x)\rangle_s = \eta_{\mu\nu} + \langle \varepsilon_{\mu\nu}(x)\rangle_\varepsilon + \tfrac{1}{4}\langle \varepsilon^\rho_\mu(x)\varepsilon_{\nu\rho}(x)\rangle_\varepsilon \tag{1.21}$$

Here, an intermediate averaging procedure $\langle\cdots\rangle_s$ should be constructed so that it will be reduced to taking the *expectation value* (1.17) of stochastic fields $\varepsilon_{\mu\nu}(x)$ in the Euclidean metric. Thus,

$$\langle \varepsilon_{\mu_1\nu_1}(x_1)\cdots\varepsilon_{\mu_n\nu_n}(x_n)\rangle_\varepsilon = 0 \qquad \text{if } n \text{ is odd}$$

and

$$\langle \varepsilon_{\mu_1\nu_1}(x_1)\cdots\varepsilon_{\mu_n\nu_n}(x_n)\rangle_\varepsilon = \sum_{\{\mu_i\nu_j\}}\prod_{i\neq j} D^{(s)}_{\mu_i\nu_i,\mu_j\nu_j}(x_i - x_j) \qquad \text{if } n \text{ is even}$$

the sum being taken over all the permutations of the indices $\{\mu_i\nu_j\}$. Second, to construct an explicit form of the function $D^{(s)}_{\mu\nu,\rho\sigma}(x)$ we use the *Euclidean postulate* (Schwinger, 1970): mapping Minkowski space-time onto the Euclidean space, the invariant vacuumlike amplitude $D^{(s)}_{\mu_i\nu_i,\mu_j\nu_j}(x_i - x_j)$ describing the full physical stochastic process preserves both its meaning and invariance character. The Euclidean postulate will turn out to be more natural if one notices that the function $D^{(s)}_{\mu\nu,\rho\sigma}(x)$ possesses all the necessary properties; the Euclidean invariant function connected with it exists everywhere including the point $x = 0$, which is just $D_{\mu\nu,\rho\sigma}(x^E)$ obtained above. To show this we define the *Fourier transform*

$$\varepsilon_{\mu\nu}(x) = (2\pi)^{-4}\int d^4p\, e^{-ipx}\tilde{\varepsilon}_{\mu\nu}(p) \tag{1.22}$$

and the *covariance*

$$\langle \tilde{\varepsilon}_{\mu\nu}(p)\tilde{\varepsilon}_{\rho\sigma}(q)\rangle_\varepsilon = i^{-1}(2\pi)^4\delta^{(4)}(p+q)\tilde{D}_{\mu\nu,\rho\sigma}(p) \tag{1.23}$$

for the stochastic field $\varepsilon_{\mu\nu}(x)$. Then

$$\langle \varepsilon_{\mu\nu}(x)\varepsilon_{\rho\sigma}(y)\rangle_\varepsilon = i^{-1}(2\pi)^{-4}\int d^4p\, e^{-ip(x-y)}\tilde{D}_{\mu\nu,\rho\sigma}(p) \tag{1.24}$$

According to the Euclidean postulate, one can carry out the substitutions

$$x^0 \to i^{-1}x_4, \qquad p^0 \to ip_4, \qquad \eta_{\mu\nu} \to \delta_{\mu\nu}$$
$$y^0 \to i^{-1}y_4$$

in expression (1.24), and it is easily seen that the obtained result coincides with (1.16) and (1.17).

It should be noted that by our construction in the given scheme, the simpler *covariance* of the type

$$D_{\mu\nu,\rho\sigma}(0) = \langle \varepsilon_{\mu\nu}(x)\varepsilon_{\rho\sigma}(x)\rangle_\varepsilon$$

will be encountered in many cases. Calculation of this type of covariance is not difficult. In particular, the expression (1.21) takes the form

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \tfrac{1}{4}D^{(s)\rho,}_{\mu,\nu\rho}(0) \tag{1.25}$$

where

$$\begin{aligned} D^{(s)\rho,}_{\mu,\nu\rho}(0) &= i^{-1}(2\pi)^{-4}\int d^4p\,[d_{\mu\nu}(p)d^\rho_\rho(p) + d_{\mu\rho}(p)d^\rho_\nu(p) \\ &\quad -\tfrac{2}{3}d^\rho_\mu(p)d_{\nu\rho}(p)]\tilde{D}^{(i)}_{l,G}(p^2) \\ &= \tfrac{5}{2}\eta_{\mu\nu}(2\pi)^{-4}\int d^4q_E \times \begin{cases} G^2q_E^4\tilde{D}^{(1)}_l(q_E^2) \\ G\tilde{D}^{(2)}_l(q_E^2) \end{cases} \\ &= \tfrac{5}{2}\eta_{\mu\nu} \times \begin{cases} G^2(2\pi)^{-4}\int d^4q_E\, q_E^4\tilde{D}^{(1)}_l(q_E^2) \\ G/l^2 \end{cases} \end{aligned}$$

Here we have turned to the Euclidean metric and used the normalization condition (1.18b).

1.3. Change of the Time Scale and Distance

By our construction, although the stochastic fluctuation in the metric (1.11) with respect to the background Minkowski space-time remains an invariant character of the velocity of light, it leads to kinematical consequences for particles moving with speed smaller than the light velocity. One of them is the change of the *time scale* for moving clocks. For the definition of this change, in our case, consider clocks moving with an arbitrary velocity in the fictitious "gravitational" field given by the stochastic metric $g^{(s)}_{\mu\nu}(x)$. Then, according to the result obtained in Section 1.1, in a coordinate system x^ν, the space-time interval between counts shown by the clocks is given by formula (1.8) with (1.11),

$$\Delta\tau_s = (-g^{(s)}_{\mu\nu}\,dx^\nu\,dx^\mu)^{1/2} \tag{1.26}$$

Since the velocity of the clocks is dx^ν/dt, the time interval between counts is defined by

$$\Delta\tau_s = dt\left(-g^{(s)}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right)^{1/2} \tag{1.27a}$$

or

$$\Delta\tau_s = dt\,(-g^{(s)}_{00})^{1/2} \tag{1.27b}$$

for the special case when the clocks are at rest. Taking into account the explicit form (1.11) and assuming that the stochastic fluctuating metric is small, we obtain a series over the field $\varepsilon_{\mu\nu}(x)$:

$$\Delta\tau_s = dt\,[A^{1/2} - \tfrac{1}{2}A^{-1/2}(\varepsilon_{\mu\nu}(x) + \tfrac{1}{4}\varepsilon^{\rho}_{\mu}(x)\varepsilon_{\nu\rho}(x))u^\mu u^\nu$$
$$-\tfrac{1}{8}A^{-3/2}\varepsilon_{\mu\nu}(x)\varepsilon_{\rho\sigma}(x)u^\mu u^\nu u^\rho u^\delta + \cdots] \tag{1.28}$$

where

$$u^\mu = dx^\mu/dt, \qquad A = -\eta_{\mu\nu}u^\mu u^\nu$$

Further, we carry out the intermediate averaging procedure as done above and calculate some tensor algebra. The result reads

$$\Delta\tau = \langle\Delta\tau_s\rangle_s = (1 - \mathrm{v}^2/c^2)^{1/2}[1 + \tfrac{5}{24}\tilde{D}(0)]\,dt \tag{1.29}$$

Here, we have used the following expressions:

$$D_{\mu\nu,\rho\sigma}(0) = \tfrac{5}{9}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \tfrac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma})\tilde{D}(0) \tag{1.30}$$

and

$$\tilde{D}(0) = \begin{cases} G^2(2\pi)^{-4}\displaystyle\int d^4q_E\, q_E^4\,\tilde{D}^{(1)}_l(q_E^2) & \text{for (1.18a)} \\ G/l^2 & \text{for (1.18b)} \end{cases} \tag{1.31}$$

Now, let us consider the element *dl* of the *spatial distance* in the space-time with stochastic metric (1.11). Due to the stochastic character of the space-time metric, the value of *dl* fluctuates and becomes of no definite length. According to the definition of "*spacelike*" *distance* in the usual theory of relativity, one can calculate a full "time" interval between leaving and coming signals at the same point of space-time with a stochastic metric, which is given by the formula (for details, see Landau and Lifschitz, 1971)

$$dx_0^{(2)} - dx_0^{(1)} = 2(-g^{(s)}_{00})^{-1}[(g^{(s)}_{0i}g^{(s)}_{0j} - g^{(s)}_{ij}g^{(s)}_{00})\,dx^i\,dx^j]^{1/2},$$
$$(i, j = 1, 2, 3) \tag{1.32}$$

According to formula (1.27b), the corresponding interval of true time $\Delta\tau_s$ is obtained by multiplication of (1.32) by the value $(-g^{(s)}_{00})^{1/2}/c$ and the distance dl between both points defined by further multiplication by $c/2$. Thus, we find

$$dl_s^2 = (g_{ij}^{(s)} - g_{0i}^{(s)} g_{0j}^{(s)} / g_{00}^{(s)})\, dx^i\, dx^j$$

In the approximation of the *weak-field limit* its averaged value acquires the form

$$dl^2 = \langle dl_s^2 \rangle_s = [\eta_{ij} + D_{0i,0j}(0) + \tfrac{1}{4} D^{\rho}_{i,j\rho}(0)]\, dx^i\, dx^j$$

Substituting here the explicit form of the functions $D_{(\ldots)}(0)$ by the formula (1.30), we get

$$dl^2 = dl_0^2 [1 + \tfrac{5}{72} \tilde{D}(0)] \tag{1.33}$$

where $\tilde{D}(0)$ is given by formula (1.31), and

$$dl_0^2 = \delta_{ij}\, dx^i\, dx^j, \qquad \eta_{ij} \equiv \delta_{ij}$$

is the *standard spatial element* of the Euclidean distance between two points. Notice that the quantity

$$\gamma_{ij}^{(s)} = g_{ij}^{(s)} - g_{0i}^{(s)} g_{0j}^{(s)} / g_{00}^{(s)} \tag{1.34}$$

is the three-dimensional metric tensor defining the metric, i.e., the geometric properties of space.

Now we turn to the problem of calculating the *red-shift* contribution due to the stochastic fluctuation of the space-time metric. For this purpose, let us consider the particular case (1.27b), when the clocks are at rest. As in the usual theory of gravity, in our case we do not observe the coefficients of change of the time scale appearing (1.27b) by measuring the time interval dt between two counts and comparing it with the averaged value $\langle \Delta\tau_s \rangle_s$. However, we can compare the coefficients of the change of the time scale due to the fluctuational nature of the space-time metric at two different points of the field. It is assumed, for example, that at point 1 we observe a light signal coming from point 2, where it appears as a result of some atomic transition. Therefore, according to formula (1.27b), the time between two successive signals arriving at point 1 will be connected with the time between those leaving from point 2 by the formula

$$dt_2 = (\Delta\tau_s)[-g_{00}(x_2)]^{-1/2}$$

If an analogous atomic transition takes place at the point 1, then the time separating the arriving light wave signals measured at point 1 is equal to

$$dt_1 = (\Delta\tau_s)[-g_{00}(x_1)]^{-1/2}$$

Thus, for the given atomic transition, the ratio of frequencies for (observing at point 1) light leaving from point 2 and light coming from point 1 is given by

$$(\nu_2/\nu_1)_s = [g_{00}(x_2)/g_{00}(x_1)]^{1/2} \tag{1.35}$$

For the limiting case of a weak field $\varepsilon_{\mu\nu}(x)$, $\nu_2/\nu_1 = 1+\Delta\nu/\nu$ and expression (1.35) takes the form

$$\begin{aligned}(\nu_2/\nu_1)_s &= 1+(\Delta\nu/\nu)_s \\ &= 1+\tfrac{1}{2}[\varepsilon_{00}(x_1)-\varepsilon_{00}(x_2)] \\ &\quad+\tfrac{1}{8}[\varepsilon_0^\rho(x_1)\varepsilon_{0\rho}(x_1)-\varepsilon_0^\rho(x_2)\varepsilon_{0\rho}(x_2)] \\ &\quad+\tfrac{1}{8}[\varepsilon_{00}^2(x_1)-\varepsilon_{00}^2(x_2)]+\tfrac{1}{4}\varepsilon_{00}^2(x_1)-\tfrac{1}{4}\varepsilon_{00}(x_2)\varepsilon_{00}(x_1)\end{aligned}$$

After the *averaging procedure,* we have

$$\Delta\nu/\nu = \langle\Delta\nu/\nu\rangle_s = \tfrac{1}{4}D_{00,00}(0) - \tfrac{1}{4}D_{00,00}(x_1-x_2) \tag{1.36}$$

where

$$D_{00,00}(0) = \tfrac{5}{6}\tilde{D}(0)$$

and

$$D_{00,00}(x_1-x_2) = (2\pi)^{-4}\int d^4p\,\Pi_{00,00}(p)\,e^{-ip(x_1-x_2)}\tilde{D}_l(p^2) \tag{1.37}$$

We first calculate $\tilde{D}(0)$ for both cases (1.18a) and (1.18b). It is easy to verify that according to the normalization condition (here and below omitting index E on the momentum variable)

$$l^2(2\pi)^{-4}\int d^4p\,\tilde{D}_l(p^2) = 1$$

for the case (1.18b) the function

$$\tilde{D}(0) = \frac{5}{6}\frac{G}{l^2} \tag{1.38}$$

for any distribution $\tilde{D}_l^{(2)}(p^2)$, i.e., it does not depend on its concrete form. For the case (1.18a), an explicit form of the distribution function $\tilde{D}_l^{(1)}(p^2)$ should be given. The choice

$$\tilde{D}^{(1)}(p^2) = c_1(1+p^2l^2)^{-5} \tag{1.39}$$

with the normalization coefficient $c_1 = 6\cdot 2^5\,\pi^2$ gives

$$\tilde{D}(0) = \frac{5}{2}\frac{G^2}{l^4} \tag{1.40}$$

for the case (1.18a). The second term in (1.36) given by formula (1.37) has a potential character and depends on the concrete realization of experimental situations. We calculate it in the static limit $p_0=0$ and in the case when the *distribution function* is given by the formula

$$\tilde{D}^{(1)}(\mathbf{p}^2)=\tfrac{1}{5}\pi\cdot 2^9 l^3(1+\mathbf{p}^2l^2)^{-5} \tag{1.41}$$

Taking into account that

$$\Pi_{00,00}(p)=\tfrac{4}{3}(p_0^2-\eta_{00}p^2)^2=\tfrac{4}{3}\mathbf{p}^4$$

we have

$$D^{(1)}_{00,00}(\mathbf{x}_1-\mathbf{x}_2)=\tfrac{4}{3}G^2(2\pi)^{-3}\int d^3p\, e^{-i\mathbf{p}(\mathbf{x}_1-\mathbf{x}_2)}\mathbf{p}^4\tilde{D}^{(1)}(\mathbf{p}^2)$$

$$=(2^8G^2l^3/15\pi^2)\int d^3p\,\mathbf{p}^4(1+\mathbf{p}^2l^2)^{-5}\,e^{-i\mathbf{p}(\mathbf{x}_1-\mathbf{x}_2)}$$

Standard integration over the angles φ and θ gives

$$D^{(1)}_{00,00}(x)=(2^8G^2l^3/15\pi^2)\int_0^\infty dp\,p^2\int_0^{2\pi}d\varphi\int_0^\pi d\theta\sin\theta \times e^{-i|p|\cdot|x|\cos\theta}(1+p^2l^2)^{-5}$$

$$=(2^{10}G^2l^3/15\pi x)\int_0^\infty dp\sin px\cdot p^5(1+p^2l^2)^{-5} \tag{1.42}$$

where $x=|\mathbf{x}_1-\mathbf{x}_2|$. After some elementary transformation this integral is reduced to the standard one

$$D^{(1)}_{00,00}(x)=\frac{2^8G^2l^3}{45\pi x}\frac{d^2}{dy^2}\left[\frac{\pi x}{16}y^{-3/2}(1+xy^{-1/2})\exp(-xy^{-1/2})\right]$$

$$=\frac{4}{45}\frac{G^2}{l^4}\left[15+15\left(\frac{x}{l}\right)-10\left(\frac{x}{l}\right)^2+\left(\frac{x}{l}\right)^3\right]e^{-x/l} \tag{1.43}$$

For the second case (1.18b) with the *distribution function*

$$\tilde{D}^{(2)}(\mathbf{p}^2)=2^5\pi l(1+\mathbf{p}^2l^2)^{-3}$$

the integral (1.42) acquires the form

$$D^{(2)}_{00,00}(x)=\frac{4}{3}\frac{G}{l^2}\left(1+\frac{x}{l}\right)e^{-x/l} \tag{1.44}$$

Collecting the results (1.38), (1.40), (1.43), and (1.44) together, we have for expression (1.36)

$$\left(\frac{\Delta\nu}{\nu}\right)_{\text{stoch}}=\left\langle\frac{\Delta\nu}{\nu}\right\rangle_s$$

$$=\frac{1}{4}\begin{cases}\frac{5}{2}\frac{G^2}{l^4}-\frac{5}{45}\frac{G^2}{l^4}e^{-x/l}\left[15+15\frac{x}{l}-10\frac{x^2}{l^2}+\left(\frac{x}{l}\right)^3\right] & \text{for (1.18a)}\\ \frac{5}{6}\frac{G}{l^2}-\frac{4}{3}\frac{G}{l^2}\left(1+\frac{x}{l}\right)e^{-x/l} & \text{for (1.18b)}\end{cases} \tag{1.45}$$

From the calculated contribution of (1.36) with (1.38) and (1.40) to the *red shift* due to the stochastic fluctuation in the metric we now find an estimation of the *fundamental length.* For this there is one interesting experimental test of the gravitational red shift, realized by Pound and Rebka (1960). They allowed a photon emitted by ^{57}Fe, due to an energy transition of 14.4 keV (0.1 mks), to fall from a height of 22.6 m, and observed its resonance absorption by the same atom ^{57}Fe. In the usual theory of gravity, if the equivalence principle is valid, one must expect that the light frequency falling into the target will be shifted by the classical value

$$(\Delta\nu/\nu)_{\text{cl}}=-\Delta\phi=\phi(x_1)|_{\text{target}}-\phi(x_2)|_{\text{source}}=2.46\times10^{-15}$$

At present, this theoretical calculation coincides with the experimental result $(\Delta\nu/\nu)_{\text{exp}}=2.6\times10^{-15}$ to an accuracy of about 1% (Pound and Snider, 1964). Therefore, the contribution due to stochastically fluctuating metric in (1.36) should be less than the experimental errors:

$$(\Delta\nu/\nu)_{\text{stoch}}\lesssim0.26\times10^{-15}$$

It is easily verified that the second term in (1.36), in accordance with (1.43), (1.44), and the condition of the experiment, is much smaller than the first term, and therefore, from the first term of the latter expression in (1.45), we conclude that

$$l\gtrsim10^{-25}\text{ cm} \tag{1.46}$$

Thus, we have shown that the change of the time scale due to the stochastic fluctuation of the space-time metric allows us to estimate the lower bound on the value of the fundamental length. However, the most stringent bound will be given in Section 7.

1.4. Appearance of an Additional Force Due to Stochastic Fluctuation in the Metric

We see that particle motion (1.5) in a fictitious "gravitational" field given by the metric (1.11) is defined by the quantities $\gamma^{\lambda}_{\mu\nu}$ in (1.6). By definition, the derivative $d^2x^\nu/d\tau^2$ is the *four-acceleration* of the particle.

Therefore, one can call the quantity $-m\gamma^{\lambda}_{\mu\nu}u^{\mu}u^{\nu}$ the four-force acting on the particle in this fictitious "gravitational" field. Now, we define this force for the constant fictitious field $\varepsilon_{\mu\nu}(x) \equiv \varepsilon_{\mu\nu}(\mathbf{x})$. According to the usual theory of relativity (for example, see Landau and Lifschitz, 1971) the necessary components of $\gamma^{\lambda}_{\mu\nu}$ in the three-dimensional case are

$$\begin{aligned}
&\gamma^{i}_{00} = \tfrac{1}{2}h^{;i} \\
&\gamma^{i}_{0j} = \frac{h}{2}(g^{i}_{;j} - g^{;i}_{j}) - \tfrac{1}{2}g_j h^{;i} \qquad (1.47) \\
&\gamma^{i}_{jk} = \lambda^{i}_{jk} + \frac{h}{2}[g_j(g^{;i}_{k} - g^{i}_{;k}) + g_k(g^{;i}_{j} - g^{i}_{;j})] + \tfrac{1}{2}g_j g_k h^{;i}
\end{aligned}$$

Here, all tensor operators (see Section 3; in particular: covariant differentiation, raising and lowering indices) are carried out in the three-dimensional space with the metric γ_{ij} of (1.34) by means of the three-dimensional vector $g^{i} = -g_{0i}/g_{00}$ and the three-dimensional scalar $h = -g_{00}$. The quantity λ^{i}_{jk} is the three-dimensional *Christoffel symbol* constructed of components of the tensor γ_{ij}, since in accordance with formula (1.6b), $\gamma^{\lambda}_{\mu\nu}$ are formed of the components of $g^{\rho\sigma}$ and $g_{\mu\nu}$, where the components of the contravariant tensor $g^{\lambda\nu}$ equal

$$g^{ij} = \gamma^{ij}, \qquad g^{0i} = g^{i} \equiv \gamma^{ij}g_j$$

Substituting (1.47) into the equation of motion

$$\frac{du^{i}}{d\tau} = -\gamma^{i}_{00}(u^{0})^2 - 2\gamma^{i}_{0j}u^{0}u^{j} - \gamma^{i}_{jk}u^{j}u^{k}$$

and making use of expressions [for details, see Section 1.5 and formula (1.64)]

$$u^{i} = (v^{i}/c)\beta, \qquad u^{0} = h^{-1/2}\beta + g_i v^{i}\beta/c, \qquad \beta = (1 - \mathbf{v}^2/c^2)^{-1/2}$$

for four-velocity in space-time with fluctuating metric and after simple transformations, we get

$$\frac{d}{d\tau}\left[\left(\frac{v^{i}}{c}\right)\beta\right] = -\tfrac{1}{2}h^{-1}h^{;i}\beta^2 - h^{1/2}\beta^2(g^{i}_{;j} - g^{;i}_{j})\frac{v^{j}}{c} - \lambda^{i}_{jk}v^{j}v^{k}\beta^2c^{-2} \qquad (1.48)$$

Acting on a particle, the potential "force" $\mathbf{f}$ is the derivative of its momentum $\mathbf{p}$ with respect to the (synchronized) proper time and is defined by the "*covariant*" *differential* in the three-dimensional space

$$f^{i}_{s} = c\beta^{-1}\frac{Dp^{i}}{D\tau} = c\beta^{-1}\frac{d}{d\tau}\beta m v^{i} + \lambda^{i}_{jk}m\beta v^{j}v^{k}$$

From (1.48) it follows that

$$\mathbf{f}_s = m\beta c^2\left\{-\text{grad}\ln\sqrt{h} + \sqrt{h}\left[\frac{\mathbf{v}}{c}\times\text{rot}\,\mathbf{g}\right]\right\} \qquad (1.49)$$

This formula coincides formally with the force acting on the particle due to the usual constant gravitational field. Notice that in our case this force has a stochastic character since the expressions h and $\mathbf{g}$ entering into it possess random properties. Now we calculate an *averaged force*. For this, we decompose it over the weak field $\varepsilon_{\mu\nu}(x)$ and carry out an averaging procedure according to the previous sections. In the *weak-field limit* we have

$$\ln\sqrt{h} = -\tfrac{1}{2}\varepsilon_{00} - \tfrac{1}{8}\varepsilon_0^\rho\varepsilon_{0\rho} - \tfrac{1}{4}\varepsilon_{00}^2 + O(\varepsilon^3)$$

$$g_i = g^i = -(\eta_{0i} + \varepsilon_{0i} + \tfrac{1}{4}\varepsilon_0^\rho\varepsilon_{i\rho})(-1 + \varepsilon_{00} + \tfrac{1}{4}\varepsilon_0^\rho\varepsilon_{0\rho})^{-1}$$
$$= \varepsilon_{0i} + \varepsilon_{0i}\varepsilon_{00} + \tfrac{1}{4}\varepsilon_0^\rho\varepsilon_{i\rho}$$

Therefore, in this approximation, the *force* (1.49) acquires the form

$$\begin{aligned} f_i = m\beta c^2 \Big\{ & \nabla_i(\tfrac{1}{2}\varepsilon_{00} + \tfrac{1}{8}\varepsilon_0^\rho\varepsilon_{0\rho} + \tfrac{1}{4}\varepsilon_{00}^2) \\ & + [1 - \tfrac{1}{2}\varepsilon_{00} - \tfrac{1}{8}(\varepsilon_0^\rho\varepsilon_{0\rho} + \varepsilon_{00}^2)] \\ & \times \frac{1}{c}\,\varepsilon_{ijk}v_j\varepsilon_{knm}\partial_n(\varepsilon_{0m} + \varepsilon_{0m}\varepsilon_{00} + \tfrac{1}{4}\varepsilon_0^\rho\varepsilon_{m\rho}) \Big\} \end{aligned} \qquad (1.50)$$

where ε_{ijk} is the *full antisymmetric tensor* of the third rank. To calculate an averaged force, one needs to modify the distribution function $\tilde{D}_l(q^2)$ in the presence of a particle with momentum $\mathbf{q}_0$ and mass m. Choose the *Gaussian* normalized *distribution*

$$\begin{aligned} (2\pi)^{-3}\int d^3q\,\tilde{D}_l^{(m)}(\mathbf{q}^2) &= 1 \\ \tilde{D}_l^{(m)}(\mathbf{q}^2) &= (2\pi)^{3/2}l^3\exp[-(\mathbf{q}-\mathbf{q}_0)^2 l^2/2] \end{aligned} \qquad (1.51)$$

and carry out an averaging procedure for an expression of the type

$$\partial_i\varepsilon_0^\rho(\mathbf{x})\cdot\varepsilon_{0\rho}(\mathbf{x})$$

by the formula

$$\langle\partial_i\varepsilon_0^\rho(\mathbf{x})\cdot\varepsilon_{0\rho}(\mathbf{x})\rangle_\varepsilon = (2\pi)^{-3}G^2\int d^3q\,q_i\tilde{D}_{0,0\rho}^\rho(\mathbf{q}) \qquad (1.52)$$

where we have used the following definitions:

$$\varepsilon_0^\rho(\mathbf{x}) = (2\pi)^{-3}\int d^3q_1\,e^{i\mathbf{q}_1\mathbf{x}}\tilde{\varepsilon}_0^\rho(\mathbf{q}_1)$$

$$\partial_i\varepsilon_{0\rho}(\mathbf{x}) = i(2\pi)^{-3}\int d^3q_2\,e^{i\mathbf{q}_2\mathbf{x}}q_{2i}\tilde{\varepsilon}_{0\rho}(\mathbf{q}_2)$$

and

$$\langle \tilde{\varepsilon}_0^\rho(\mathbf{q}_1)\varepsilon_{0\rho}(\mathbf{q}_2)\rangle_\varepsilon = i^{-1}(2\pi)^3\delta^{(3)}(\mathbf{q}_1+\mathbf{q}_2)\tilde{D}^\rho_{0,0\rho}(\mathbf{q}_1) \tag{1.53}$$

It is just a modified version of the *covariance* (1.23) for the three-dimensional case. Then, from (1.50) we find

$$\begin{aligned}\langle -\nabla_i \ln \sqrt{h}\rangle_s &= \tfrac{1}{4}\langle \partial_i \varepsilon_0^\rho(\mathbf{x})\cdot\varepsilon_{0\rho}(\mathbf{x})\rangle + \tfrac{1}{2}\langle \partial_i \varepsilon_{00}(\mathbf{x})\cdot\varepsilon_{00}(\mathbf{x})\rangle \\ &= -\tfrac{1}{6}G^2(2\pi)^{-3}\int d^3q\, \mathbf{q}^4 q_i \tilde{D}_l^{(m)}(\mathbf{q}^2)\end{aligned} \tag{1.54}$$

Here, the factor $\mathbf{q}^4$ results from the identities

$$\Delta^\rho_{0,0\rho}(q)|_{q_0\to 0} = [d_0^\rho(q)d_{0\rho}(q) + d_\rho^\rho(q)d_{00}(q) - \tfrac{2}{3}d_0^\rho d_{0\rho}]|_{q_0\to 0} = -\tfrac{10}{3}\mathbf{q}^4$$

and

$$\Delta_{00,00}(q)|_{q_0\to 0} = \tfrac{4}{3}\mathbf{q}^4$$

for the divisor $d^{(1)}_{\mu\nu}(q) = q_\nu q_\mu - q^2\eta_{\mu\nu}$ and distributions of the type of (1.18a). On the other hand, it is easy to verify that the second averaged term in (1.49) or (1.50) becomes zero by the construction of tensor structures

$$\Delta_{00,0i}(q)|_{q_0\to 0} = \Delta^\rho_{0,i\rho}(q)|_{q_0\to 0} = 0$$

Thus, an averaged force (1.49) is determined by the formula (1.54), the calculation of which is not difficult for a concrete form of the distributions $D_l^{(m)}(q^2)$. For the Gaussian distribution it takes the form

$$f_i = \langle f_i^s\rangle_s = -m\beta c^2 \tfrac{1}{6}G^2 l^{-4} p_i[35 + (pl)^4 + 14(pl)^2], \qquad p = \sqrt{\mathbf{p}^2} \tag{1.55}$$

The *equation of motion* in the nonrelativistic limit becomes

$$\frac{d\mathbf{v}}{dt} = -\tfrac{1}{6}G^2 l^{-4} mc\mathbf{v}[35 + (mvl)^4 + 14(mvl)^2] \tag{1.56}$$

where the dimension of the particle's mass m ($\lambda = \hbar/mc$) is expressed as a length, i.e., $[m] = [\mathrm{cm}^{-1}]$. We see that finding the particle's trajectory is complicated and is reduced to the solution of an essential nonlinear differential equation of first order. However, the case (1.18b) is very simple, for which we have

$$f_i = -m\beta c^2 \tfrac{1}{6} p_i G l^{-2} \tag{1.57}$$

or

$$\frac{d\mathbf{v}}{dt} = -\tfrac{1}{6} mcGl^{-2}\mathbf{v}$$

The latter gives

$$\mathbf{v}(t) = \mathbf{v}_0 \exp(-\tfrac{1}{6}Gl^2 mct) \tag{1.58}$$

Thus, in this particular case, the stochastic fluctuation of the space-time metric gives rise to a friction force in the dynamical motion of a particle.

It should be noted that instead of a minus sign in the definition of the force (1.55), it is quite possible to take a plus sign. This uncertainty in the choice of sign for the force due to a stochastic metric is caused by the two possible equivalent definitions of the factor $\exp(\pm i\mathbf{qx})$ for the *Fourier transform* of the stochastic field $\varepsilon_{\mu\nu}(\mathbf{x})$. This problem will be discussed in Section 7.

1.5. Particle Motion in Fictitious "Gravitational" Field

1.5.1. Four-Velocity

When there is no external force acting on a particle, its equation of motion in the fictitious "gravitational" field with metric (1.11) is defined by formula (1.5). By means of the concept of covariant differentiation defined below (Section 3), this equation may be rewritten as

$$\frac{Du^\lambda}{D\tau} \equiv \frac{du^\lambda}{d\tau} + \gamma^\lambda_{\mu\nu} u^\mu u^\nu = 0 \tag{1.59}$$

where $\gamma^\lambda_{\mu\nu}$ is the modified affine-connection (1.16a) and

$$u^\lambda \equiv \frac{dx^\lambda}{c\,d\tau} \tag{1.60}$$

is the *four-vector of velocity*. The components of the four-velocity depend on each other since $d\tau^2 = -g^{(s)}_{\mu\nu}\,dx^\mu\,dx^\nu$, and therefore

$$u_s^2 = g^{(s)}_{\mu\nu} u^\mu u^\nu = -1 \tag{1.61}$$

Geometrically, this means that in space-time with a stochastic metric, u^μ is also a *unit vector*. By analogy with the definition of the four-velocity, we call the second derivative

$$\frac{d^2x^\nu}{c^2\,d\tau^2} = \frac{du^\mu}{c\,d\tau}$$

the *four-acceleration*. Differentiating relation (1.61) with respect to the proper time $c\,d\tau$, we find

$$g^{(s)}_{\mu\nu} u^\mu \frac{du^\nu}{c\,d\tau} = u_\nu \frac{du^\nu}{c\,d\tau} = 0 \tag{1.62}$$

i.e., in the fluctuating space-time, the four-vectors of velocity and acceleration are "mutually perpendicular" (here and below we omit the index s any stochastic quantities).

It is interesting to define four-velocity for the particular case when the stochastic field $\varepsilon_{\mu\nu}(x)$ does not depend on the time variable x^0; we call it world time. If a particle leaves from point A at world time x^0 and arrives at point B in the neighborhood of A at the moment x^0+dx^0, then for the definition of velocity one needs to take not the time interval $(x^0+dx^0)-x^0=dx^0$, but the difference between x^0+dx^0 and the moment $x^0-(g_{0i}/g_{00})\,dx^i$, which is the same time x^0 at the point B as well as at A:

$$(x^0+dx^0)-(x^0-g_{0i}\,dx^i/g_{00})=dx^0+g_{0i}\,dx^i/g_{00}$$

Multiplying this by the factor $(-g_{00})^{1/2}/c$, we obtain the corresponding interval of the proper time, so that *velocity* is

$$v^i=c\,dx^i[h(dx^0-g_i\,dx^i)]^{-1/2}$$

where we have used the notation

$$h=-g_{00}, \qquad g_i=-g_{0i}/g_{00} \tag{1.63}$$

Notice that for such a definition, the interval $ds=c\,d\tau$ is expressed through velocity in the usual form

$$\begin{aligned} ds^2&=-g_{00}(dx^0)^2-2g_{0i}\,dx^0\,dx^i-g_{ij}\,dx^i\,dx^j \\ &=h(dx^0-g_i\,dx^i)^2-dl^2=h(dx^0-g_i\,dx^i)^2(1-\mathbf{v}^2/c^2) \end{aligned}$$

Moreover, $\mathbf{v}^2$ needs to be understood as the square of a three-vector in the space with metric tensor γ_{ij}, (1.34):

$$\mathbf{v}^2=v_iv^i, \qquad v_i=\gamma_{ij}v^j$$

The components of the four-velocity $u^i=dx^i/ds$ are equal to

$$\begin{aligned} u^i&=(v^i/c)(1-\mathbf{v}^2/c^2)^{-1/2} \\ u^0&=h^{-1/2}(1-\mathbf{v}^2/c^2)^{-1/2}+(g_iv^i/c)(1-\mathbf{v}^2/c^2)^{-1/2} \end{aligned} \tag{1.64}$$

1.5.2. Four-Force

As in the case of relativistic mechanics, we define the *four-force* acting on a particle with the coordinates $x^\nu(\tau)$ in the space-time with a fluctuating metric by the formula

$$f^\lambda=m\frac{d^2x^\lambda}{c^2\,d\tau^2}+m\gamma^\lambda_{\mu\nu}u^\mu u^\nu \tag{1.65}$$

Obviously, if f^μ is known, then one can calculate the motion of a particle. Now we link this force with the usual force F^α defined in the *local inertial system of reference* ξ^α free from the fictitious "gravitational" field $\varepsilon_{\mu\nu}(x)$.

Note that upon the passage from the ξ^α to the x^ν system of reference the differential of coordinates transforms in the standard way

$$dx^\mu = \frac{\partial x^\mu}{\partial \xi^\alpha}\, d\xi^\alpha$$

while $d\tau$ is invariant. Therefore, from (1.65) it follows that the rule of transformation for the quantity f^μ acquires the form

$$f^\mu = (\partial x^\mu / \partial \xi^\alpha) F^\alpha \tag{1.66}$$

Any quantity, such as dx^α and f^α, transformed by rule (1.66) is called a four-vector (for details, see Section 3).

It is well known that in accordance with the special theory of relativity, the usual *relativistic force* F^α is connected with the Newtonian force $\mathbf{f}_N$ ($f_N^0 = 0$) by the formula

$$\mathbf{F} = \mathbf{f}_N + (\beta - 1)\mathbf{v}(\mathbf{v}\cdot\mathbf{f}_N)/\mathbf{v}^2$$

$$F^0 = \beta(\mathbf{v}\cdot\mathbf{f}_N) = \mathbf{v}\cdot\mathbf{F}$$

where

$$\beta = (1 - \mathbf{v}^2)^{-1/2} \qquad (c = 1)$$

In space-time with a stochastic fluctuating metric the Jacobian of transformation between coordinates ξ^α and x^μ is given by (1.12), and therefore, the force (1.66) is easily found by means of the known usual force F^α. Notice that from condition (1.61) or (1.62) it follows that

$$2 g_{\mu\nu}^{(s)} f^\mu \frac{dx^\nu}{d\tau} = 0 \tag{1.67}$$

To show that it does indeed hold, we remark that the right-hand side of (1.67) is invariant under the transformation between the coordinates ξ^α and x^μ, i.e.,

$$g_{\mu\nu}^{(s)} f'^\mu \frac{dx'^\nu}{d\tau} = g_{\mu\nu}^{(s)} \frac{\partial x^\mu}{\partial \xi^\alpha} F^\alpha \frac{\partial x^\nu}{\partial \xi^\beta} \frac{d\xi^\beta}{d\tau}$$

Making use of the definitions (1.4) and (1.11) for the Jacobian of transformation $\partial x^\lambda / \partial \xi^\alpha$ and the stochastic metric $g_{\mu\nu}^{(s)}$, we get

$$g_{\mu\nu}^{(s)} f'^\mu \frac{dx'^\nu}{d\tau} = \eta_{\rho\sigma} \frac{\partial \xi^\rho}{\partial x^\mu} \frac{\partial \xi^\sigma}{\partial x^\nu} \frac{\partial x^\mu}{\partial \xi^\alpha} \frac{\partial x^\nu}{\partial \xi^\beta} F^\alpha \frac{d\xi^\beta}{d\tau}$$

$$= \eta_{\alpha\beta} F^\alpha \frac{d\xi^\beta}{d\tau}$$

The latter equals zero in the system of reference in which a particle is at rest. Indeed, in accordance with the usual special theory of relativity, if the particle is at rest at the given time moment, then proper time $d\tau$ coincides with dt, so that $F^i = f^i_N$, where f^i_N are the Cartesian components of the nonrelativistic force **F** and $F^0 = 0$.

1.5.3. Energy, Momentum, and an Additional Potential

The *four-momentum* of the particle in our fictitious "gravitational" field is defined as

$$\mathcal{P}^\mu = mcu^\mu$$

and its square equals (omitting the symbol s on the metric tensor $g_{\mu\nu}$)

$$g_{\mu\nu}\mathcal{P}^\mu\mathcal{P}^\nu = \mathcal{P}_\mu\mathcal{P}^\mu = -m^2c^2 \tag{1.68}$$

Instead of $\mathcal{P}_\mu$, substituting $\partial S/\partial x^\mu$ into (1.68), we find the *Hamilton-Jacobi equation* for the particle in this "gravitational" field from the expression

$$g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + m^2c^2 = 0 \tag{1.69}$$

We observe that due to the stochastic fluctuation of the space-time metric, in the limiting case, when the velocity of the particle is small, an *additional* nonrelativistic "*potential*" also appears. To connect this fictitious "potential" with the metric tensor $g_{\mu\nu}$ we act in the same way as in the case of the usual theory of gravity (Landau and Lifschitz, 1971). Let

$$L = -mc^2 + \tfrac{1}{2}m\mathbf{v}^2 - m\varphi_f$$

be the *Lagrangian function* of the nonrelativistic particle in our fictitious "gravitational" field. The *nonrelativistic action* of the particle in it has the form

$$S = \int L\,dt = -mc(c - \tfrac{1}{2}\mathbf{v}^2/c + \varphi_f/c)\,dt$$

Comparing this with the expression $S = -mc\int ds$, $ds = c\,d\tau$, we see that

$$ds = (c - \tfrac{1}{2}\mathbf{v}^2/c + \varphi_f/c)\,dt$$

Taking the square and omitting terms going to zero at the limit $c \to \infty$, we find

$$ds^2 = (c^2 + 2\varphi_f)\,dt^2 - d\mathbf{r}^2 \tag{1.70}$$

where we have used the equality $\mathbf{v}\,dt = d\mathbf{r}$.

Thus, the component of the metric tensor g_{00} in this limiting case takes the form

$$g_{00} = -1 - 2\varphi_f/c^2 \tag{1.71}$$

From (1.71) it is easily seen that other components of $g_{\mu\nu}$ are

$$g_{ij} = \delta_{ij}, \qquad g_{0i} = 0$$

Thus, in the nonrelativistic case, the stochastic fluctuation of the space-time metric gives rise to the appearance of an additional "potential"

$$\varphi_f^{(1)} = \tfrac{1}{2}c^2(-1 - g_{00}) \tag{1.72}$$

and, in accordance with formula (1.11), its averaged value is constant,

$$\langle\varphi_f^{(1)}\rangle_s = \tfrac{5}{16}c^2\tilde{D}(0), \qquad \eta_{00} = -1 \tag{1.73}$$

everywhere. If we take into account the next term in the approximation, we have

$$\varphi_f^{(2)} = \tfrac{1}{2}c^2[-\varepsilon_{00}(x) - \tfrac{1}{4}\varepsilon_0^\rho(x)\varepsilon_{0\rho}(x) - \tfrac{1}{2}\varepsilon_{00}^2(x)]$$

In particular, when the particle moves in the constant fictitious field $\varepsilon_{\mu\nu}(\mathbf{x})$ its *energy* is defined as the derivative $(-c\,\partial S/\partial x^0)$ of the action S with respect to the world time x^0. For example, it follows from this that x^0 does not enter into the Hamilton–Jacobi equation explicitly. Defined in this way, the energy $\mathscr{E}$ is the time component of the covariant four-vector of momentum $p_\mu = mcu_\mu = mcg_{\mu\nu}u^\nu$. In a static field, $ds^2 = -g_{00}(dx^0)^2 - dl^2$, and therefore,

$$\mathscr{E}_0 = -mc^2 g_{00}\frac{dx^0}{ds} = -mc^2 g_{00}\frac{dx^0}{(-g_{00}\,dx_0^2 - dl^2)^{1/2}}$$

We introduce the *velocity*

$$v = \frac{dl}{d\tau} = \frac{c\,dl}{(-g_{00}\,dx_0^2)^{1/2}}$$

of the particle measured by the proper time, i.e., by an observer located at a given place. Then, for *energy* we get

$$\mathscr{E}_0 = mc^2(1 - \mathbf{v}^2/c^2)^{-1/2}(-g_{00})^{1/2} \tag{1.74}$$

This is simply the quantity which remains unchanged upon the particle motion in the constant fictitious field $\varepsilon_{\mu\nu}(\mathbf{x})$.

On the other hand, by using the definition of velocity (1.64) for a particle moving in a stationary field, it can be easily verified that the expression

$$\mathscr{E}_0 = -mc^2 g_{0i}u^i = mc^2 h(u^0 - g_i u^i)$$

after substituting (1.64) into it, gives the form (1.74), as expected. Finally, the averaged energy in our scheme becomes

$$\bar{\mathscr{E}} = \langle\mathscr{E}_0\rangle_s = mc^2(1 - \mathbf{v}^2/c^2)^{-1/2}[1 + \tfrac{5}{24}\tilde{D}(0)] \tag{1.75}$$

The latter may be understood as a *change of the particle mass* $m \to M = m + \delta m$ in the space-time with a stochastic metric:

$$\bar{\mathscr{E}} = Mc^2(1 - \mathbf{v}^2/c^2)^{-1/2}, \qquad M = m + \delta m, \qquad \delta m = \tfrac{5}{24} m\tilde{D}(0) \quad (1.76)$$

It is important to notice that if from the very beginning we use the *Euclidean postulate*, then the correction of $\langle \varepsilon_{\mu\nu}(x)\varepsilon_{\rho\delta}(x)\rangle$ is constructed immediately in the Euclidean metric. Then, the *covariance* (1.30) acquires the form

$$D^E_{\mu\nu,\rho\sigma}(0) = \tfrac{5}{9}(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} - \tfrac{1}{2}\delta_{\mu\nu}\delta_{\rho\sigma})\tilde{D}(0) \quad (1.77)$$

where $\delta_{\mu\nu}$ is the *Euclidean metric*:

$$\delta_{\mu\nu} = \begin{cases} 0 & \text{if} \quad \mu \neq \nu \\ 1 & \text{if} \quad \mu = \nu \end{cases}$$

In this case, the corresponding sign in expression (1.73) and in the second term in (1.75) is changed conversely; the result reads

$$\langle \varphi_f \rangle_s = -\tfrac{5}{16}\tilde{D}(0)c^2$$

and

$$\bar{\mathscr{E}} = mc^2(1 - \mathbf{v}^2/c^2)^{-1/2}[1 - \tfrac{5}{12}\tilde{D}(0)] \quad (1.78)$$

or

$$\bar{\mathscr{E}} = Mc^2(1 - \mathbf{v}^2/c^2)^{-1/2}, \qquad M = m - \delta m$$

where

$$\delta m = \tfrac{5}{12} m\tilde{D}(0) \quad (1.79)$$

Finally, it should be noted that in accordance with the Euclidean postulate, an expression of the type $\varepsilon^{\rho}_{\mu}(x)\varepsilon_{\nu\rho}(x)u^{\mu}u^{\nu}$ is transformed to $\varepsilon^{\rho}_{\mu}(x^E)\varepsilon_{\nu\rho}(x^E)u^{\mu}_E u^{\nu}_E$, $u^{\mu}_E = (u^4_E = iu^0, \mathbf{u}_E = \mathbf{u})$, and, therefore,

$$\langle \varepsilon^{\rho}_{\mu}(x^E)\varepsilon_{\nu\rho}(x^E)\rangle_{\varepsilon} u^{\mu}_E u^{\nu}_E = \tfrac{5}{2}\delta_{\mu\nu}u^{\mu}_E u^{\nu}_E \Rightarrow \tfrac{5}{2}(-u_0^2 + \mathbf{u}^2)$$

which coincides with covariance $\langle \varepsilon^{\rho}_{\mu}(x)\varepsilon_{\nu\rho}(x)\rangle_{\varepsilon} u^{\mu}u^{\nu}$ obtained above. For this reason, the expression (1.29) does not change and its invariant properties remain in both the Euclidean and pseudo-Euclidean descriptions of stochastic processes.

2. THE GENERAL EQUIVALENCE PRINCIPLE IN SPACE-TIME WITH STOCHASTIC METRIC

2.1. Reformulation of the Equivalence Principle

Now we consider an external gravitational field and attempt to reconstruct the general theory of gravity from the point of view of the stochastic fluctuation of the space-time metric. In this case, our correspondence principle says that when the external gravitational field is absent, then the modified general theory of gravity, expounded below, should become the special theory of relativity with the stochastic metric reconstructed above. It turns out that successful reconstruction of the expected theory is possible if we use the equivalence principle with respect to the system of reference x^μ with stochastic metric (1.11).

It is well known that the *equivalence principle* between gravity and inertia can be understood as the reaction of a physical system on the external gravitational field. It is asserted that no external static homogeneous gravitational field whatever can be detected in a freely falling elevator, since in this field an observer, test body, and the elevator itself acquire the same acceleration. Following Weinberg (1972), one can easily prove this for an N-particle system moving with nonrelativistic velocity under an action force (for example, electromagnetic and gravitational) $\mathbf{f}(\mathbf{x}_n - \mathbf{x}_m)$ in the external gravitational field. The *equation of motion* is

$$m_n \, d^2\mathbf{x}_n/dt^2 = m_n\mathbf{g} + \sum_k \mathbf{f}(\mathbf{x}_n - \mathbf{x}_k), \qquad n, k = 1, 2, \ldots, N \tag{2.1}$$

Assuming the following *non-Galilean transformation* of space-time coordinates

$$\mathbf{x}' = \mathbf{x} - \tfrac{1}{2}\mathbf{g}t^2, \qquad t' = t \tag{2.2}$$

one finds that the term with $\mathbf{g}$ is compensated by the inertial "force" and the equation of motion takes the form

$$m_n \, d^2\mathbf{x}'_n/dt'^2 = \sum_k f(\mathbf{x}'_n - \mathbf{x}'_k) \tag{2.3}$$

Therefore, an observer O using coordinates $\mathbf{x}$, t and a freely falling colleague O' using coordinates $\mathbf{x}'$, t' do not find any difference in the laws of mechanics, with the exception that O will observe the influence of a gravitational field where O' will not.

However, in our case, both observers are under action due to an additional fictitious "gravitational" field with stochastic metric $g^{(s)}_{\mu\nu}$. This fact requires redefinition of the concept of inertia or an inertial system of reference. Under an inertial system of reference we understand a system of reference in which a fictitious "gravitational" field is always present. We call this system of reference the quasilocal-inertial system of reference. Thus, in our scheme, the generalized equivalence principle as formally formulated is based on the assumption that at every point of space-time in an arbitrarily chosen gravitational field (not only a static one) one can choose "the quasilocal-inertial" system of reference x^μ (with stochastic metric) such that in a sufficiently small neighborhood of the given point, the laws of nature will have the same form as in the nonaccelerated Cartesian system of reference. The equivalence principle thus formulated will be called the generalized equivalence principle or the *equivalence principle of the second level.*

2.2. Gravitational Force and Stochastic Metric

Let us consider a "freely" moving particle under the action of purely gravitational forces. Here, the word "freely" means that an additional stochastic radiation field $\varepsilon_{\mu\nu}(x)$ acting on the particle is present everywhere, which violates slightly the usual equivalence principle with respect to the (pseudo-) Riemannian system of coordinates. To obtain a general form of the equation of motion of the particle in the presence of arbitrary gravitational fields (including the fictitious stochastic background field $\varepsilon_{\mu\nu}$), we formally consider a freely falling system of reference ξ^α in which a particle moves along a rectilinear trajectory given by the equation

$$d^2\xi^\alpha / d\tau^2 = 0 \tag{2.4}$$

where

$$d\tau^2 = -\eta_{\alpha\beta}\, d\xi^\alpha\, d\xi^\beta \tag{2.5}$$

is the *proper time.* Notice that, according to the equivalence principle of the first level formulated in Section 1, from equations (2.4) and (2.5) we have obtained the corresponding equations (1.5) and (1.8) for the fictitious stochastic field $\varepsilon_{\mu\nu}(x)$ with the stochastic metric (1.9) given by formula (1.11). Now assume that we take any other system of reference z^μ, which may be the (pseudo-) Riemannian system of coordinates resting with respect to the laboratory system and a curvilinear, accelerated, rotating, or any other system of reference at our desire. In this case, the coordinates ξ^α (or x^ν) of a freely (or "almost freely") falling system of reference are a function of z^λ, and equation (2.4) acquires the form

$$\frac{d^2 z^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} = 0 \tag{2.6}$$

by analogy with the equation of motion (1.5) of the particle moving in the fictitious stochastic field $\varepsilon_{\mu\nu}(x)$ only. By our *correspondence principle,* if external gravitational fields are absent, equation (2.6) turns into (1.5). In this case we must put $z^\nu \equiv x^\nu$. It is assumed that the connection between them in the presence of the external gravitational fields [with the exception of the fictitious stochastic background field $\varepsilon_{\mu\nu}(x)$] will be defined in a usual form as in the linearized theory of gravity. In (2.6) the function $\Gamma^\lambda_{\mu\nu}$

is just the *affine connection* defined by

$$\Gamma^{\lambda}_{\mu\nu} \equiv \frac{\partial z^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial z^{\mu}\, \partial z^{\nu}} \tag{2.7}$$

It is obvious that in our case the well-known *multiplication rules*

$$\frac{\partial x^{\alpha}}{\partial z^{\mu}} \frac{\partial z^{\lambda}}{\partial x^{\alpha}} = \frac{\partial z^{\beta}}{\partial x^{\mu}} \frac{\partial x^{\lambda}}{\partial z^{\beta}} = \delta^{\lambda}_{\mu} \tag{2.8}$$

or

$$\frac{\partial \xi^{\alpha}}{\partial z^{\mu}} \frac{\partial z^{\lambda}}{\partial \xi^{\alpha}} = \frac{\partial z^{\beta}}{\partial \xi^{\mu}} \frac{\partial \xi^{\lambda}}{\partial z^{\beta}} = \delta^{\lambda}_{\mu} \tag{2.9}$$

are valid. The *proper time* (2.5) can also be written in an arbitrary (stochastic) system of reference:

$$d\tau^2 = -\eta_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial z^{\mu}}\, dz^{\mu} \frac{\partial \xi^{\beta}}{\partial z^{\nu}}\, dz^{\nu} \tag{2.10}$$

or [using the definition (1.11)]

$$d\tau^2 = -G_{\mu\nu}\, dz^{\mu}\, dz^{\nu} \tag{2.11}$$

where $G_{\mu\nu}$ is the true metric tensor defined by the formula

$$\begin{aligned} G_{\mu\nu} &\equiv \frac{\partial \xi^{\rho}}{\partial z^{\mu}} \frac{\partial \xi^{\delta}}{\partial z^{\nu}} \eta_{\rho\delta} = \frac{\partial \xi^{\rho}}{\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial z^{\mu}} \frac{\partial \xi^{\delta}}{\partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial z^{\nu}} \eta_{\rho\delta} \\ &= g^{(s)}_{\alpha\beta}(x) \frac{\partial x^{\alpha}}{\partial z^{\mu}} \frac{\partial x^{\beta}}{\partial z^{\nu}} \end{aligned} \tag{2.12}$$

Here $g^{(s)}_{\alpha\beta}(x)$ is given by formula (1.11). Taking into account (1.11) and natural transformations of the tensor quantities

$$\varepsilon'_{\mu\nu} = \varepsilon_{\mu\nu}(z) = \varepsilon_{\alpha\beta}(x) \frac{\partial x^{\alpha}}{\partial z^{\mu}} \frac{\partial x^{\beta}}{\partial z^{\nu}}, \qquad \varepsilon^{\alpha}_{\mu}(z) = \varepsilon^{\alpha}_{\rho}(x) \frac{\partial x^{\rho}}{\partial z^{\mu}}$$

and

$$\varepsilon_{\alpha\nu}(z) = \varepsilon_{\delta\alpha} \frac{\partial x^{\delta}}{\partial z^{\nu}}$$

from (2.12), we immediately get

$$G_{\mu\nu}(z) = \overset{0}{g}_{\mu\nu}(z) + \varepsilon_{\mu\nu}(z) + \tfrac{1}{4}\varepsilon^{\rho}_{\mu}(z)\varepsilon_{\nu\rho}(z) \tag{2.13}$$

where

$$\overset{0}{g}_{\mu\nu}(z) = \eta_{\alpha\beta} \frac{\partial x^{\alpha}}{\partial z^{\mu}} \frac{\partial x^{\beta}}{\partial z^{\nu}} \tag{2.14}$$

is caused by a purely external gravitational field and turns into $\eta_{\mu\nu}$ when the latter is absent ($z^{\nu} \equiv x^{\nu}$). The last two terms in (2.13) result from the stochastic fluctuational properties of the space-time metric.

For the *photon and neutrino* the *equation of motion* in the "freely" falling system of reference has the same form as (2.4) with the exception that their proper time (2.5) is not independent, since for these particles the right-hand side of (2.5) vanishes. Instead of τ, one uses $\sigma \equiv \xi^0$, so that equations (2.4) and (2.5) take the form

$$\frac{d^2\xi^\alpha}{d\sigma^2}=0$$

$$-\eta_{\alpha\beta}\frac{d\xi^\alpha}{d\sigma}\frac{d\xi^\beta}{d\sigma}=0$$

as in the usual case. By the same method as above, the motion in the system of reference with a stochastic metric in an arbitrary gravitational field reads

$$\frac{d^2z^\mu}{d\sigma^2}+\Gamma^\mu_{\nu\lambda}\frac{dz^\nu}{d\sigma}\frac{dz^\lambda}{d\sigma}=0 \tag{2.15}$$

$$-G_{\mu\nu}\frac{dz^\mu}{d\sigma}\frac{dz^\nu}{d\sigma}=0 \tag{2.16}$$

where $\Gamma^\mu_{\nu\lambda}(z)$ and $G_{\mu\nu}$ are expressed by the same formulas (2.7) and (2.13), respectively.

2.3. Connection Between $G_{\mu\nu}(z)$ and $\Gamma^\lambda_{\mu\nu}(z)$

As is shown above, in space-time with a stochastic metric the field defining the gravitational force is expressed through the "affine connection" $\Gamma^\lambda_{\mu\nu}(z)$, whereas the proper time interval is given by the metric tensor $G_{\mu\nu}(z)$. Now we show that $G_{\mu\nu}(z)$ is also the gravitational potential, i.e., its derivative gives the field $\Gamma^\lambda_{\mu\nu}(z)$. Notice that the connection obtained below formula (2.24) also preserves its form for the quantities defined by formulas (1.6a) and (1.9); it is just (1.6b).

We recall that the metric tensor is given by the first term of expression (2.12),

$$G_{\mu\nu}=\eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial z^\mu}\frac{\partial\xi^\beta}{\partial z^\nu}$$

Differentiation of this term with respect to z^λ yields

$$\frac{\partial G_{\mu\nu}}{\partial z^\lambda}=\frac{\partial^2\xi^\alpha}{\partial z^\lambda\,\partial z^\mu}\frac{\partial\xi^\beta}{\partial z^\nu}\eta_{\alpha\beta}+\frac{\partial\xi^\alpha}{\partial z^\mu}\frac{\partial^2\xi^\beta}{\partial z^\lambda\,\partial z^\nu}\eta_{\alpha\beta} \tag{2.17}$$

Further, multiplying equation (2.7) by the Jacobian $\partial\xi^\beta/\partial z^\lambda$ and making use of the multiplication rule (2.8), we get the following equation for ξ^α:

$$\frac{\partial^2\xi^\alpha}{\partial z^\mu\,\partial z^\nu}=\Gamma^\lambda_{\mu\nu}\frac{\partial\xi^\alpha}{\partial z^\lambda} \tag{2.18}$$

Substituting (2.18) into (2.17), we find

$$\frac{\partial G_{\mu\nu}}{\partial z^\lambda}=\Gamma^\rho_{\lambda\mu}\frac{\partial\xi^\alpha}{\partial z^\rho}\frac{\partial\xi^\beta}{\partial z^\nu}\eta_{\alpha\beta}+\Gamma^\rho_{\lambda\nu}\frac{\partial\xi^\alpha}{\partial z^\mu}\frac{\partial\xi^\beta}{\partial z^\rho}\eta_{\alpha\beta}$$

With definition (2.12) this equality takes the form

$$\frac{\partial G_{\mu\nu}}{\partial z^\lambda}=\Gamma^\rho_{\lambda\mu}G_{\rho\nu}+\Gamma^\rho_{\lambda\nu}G_{\rho\mu} \tag{2.19}$$

To express the generalized affine connection $\Gamma^\lambda_{\mu\nu}(z)$ through the metric tensor $G_{\mu\nu}(z)$, we add to (2.19) an analogous relation with rearranged indices μ and λ and subtract from (2.19) an analogous relation with rearranged indices ν and λ. As a result, we get

$$\begin{aligned}\frac{\partial G_{\mu\nu}}{\partial z^\lambda}+\frac{\partial G_{\lambda\nu}}{\partial z^\mu}-\frac{\partial G_{\mu\lambda}}{\partial z^\nu}&=G_{\delta\nu}\Gamma^\delta_{\lambda\mu}+G_{\delta\mu}\Gamma^\delta_{\lambda\nu}+G_{\delta\nu}\Gamma^\delta_{\mu\lambda}+G_{\delta\lambda}\Gamma^\delta_{\mu\nu}\\&\quad-G_{\delta\lambda}\Gamma^\delta_{\nu\mu}-G_{\delta\mu}\Gamma^\delta_{\nu\lambda}\\&=2G_{\delta\nu}\Gamma^\delta_{\lambda\mu}\end{aligned} \tag{2.20}$$

where we have taken into account the fact that $\Gamma^\delta_{\mu\nu}$ and $G_{\mu\nu}$ are symmetric under the rearrangement of indices μ and ν.

Further, one needs to define an *inverse tensor* $G^{\nu\sigma}$ with respect to $G_{\nu\sigma}$, i.e.,

$$G^{\nu\sigma}G_{\delta\nu}=\delta^\sigma_\delta \tag{2.21}$$

It should be noted that definition (2.12) ensures the existence of the *inverse tensor*

$$\begin{aligned}G^{\nu\sigma}\equiv G^{\sigma\nu}&\equiv g^{\alpha\beta}_{(s)}\frac{\partial z^\nu}{\partial x^\alpha}\frac{\partial z^\sigma}{\partial x^\beta}\\&=g_0^{\nu\sigma}-\varepsilon^{\nu\sigma}(z)+\tfrac{3}{4}\varepsilon^{\lambda\sigma}(z)\varepsilon^\nu_\lambda(z)+O(\varepsilon^3)\end{aligned} \tag{2.22}$$

Indeed, making use of the well-known multiplication rule (2.8) and relation (1.13b) we find

$$\begin{aligned}G^{\nu\sigma}G_{\delta\nu}&=g^{\alpha\beta}_{(s)}\frac{\partial z^\nu}{\partial x^\alpha}\frac{\partial z^\sigma}{\partial x^\beta}\cdot g^{(s)}_{\gamma\rho}\frac{\partial x^\gamma}{\partial z^\delta}\frac{\partial x^\rho}{\partial z^\nu}\\&=g^{\alpha\beta}_{(s)}\frac{\partial z^\sigma}{\partial x^\beta}g^{(s)}_{\gamma\alpha}\frac{\partial x^\gamma}{\partial z^\delta}\\&=\frac{\partial z^\sigma}{\partial x^\beta}\frac{\partial x^\beta}{\partial z^\delta}=\delta^\sigma_\delta\end{aligned} \tag{2.23}$$

which coincides with condition (2.21).

Now we return to equation (2.20), multiply it by $G^{\nu\sigma}$, and find

$$\Gamma^{\sigma}_{\lambda\mu}(z)=\tfrac{1}{2}G^{\nu\sigma}_{(z)}\left\{\frac{\partial G_{\mu\nu}(z)}{\partial z^{\lambda}}+\frac{\partial G_{\lambda\nu}(z)}{\partial z^{\mu}}-\frac{\partial G_{\mu\lambda}(z)}{\partial z^{\nu}}\right\} \tag{2.24}$$

Sometimes, the left-hand side of an expression of the type (2.24) is called a *Christoffel symbol* and is denoted by $\{^{\sigma}_{\lambda\mu}\}$. In our case we call it the generalized Christoffel symbol. The relation between $\Gamma^{\sigma}_{\lambda\mu}$ and $G_{\mu\nu}$ allows us to obtain an interesting consequence of the theory, which asserts that the equation of motion of an "almost freely" falling particle in space-time with a stochastic metric automatically preserves the form of the proper time interval $d\tau^2$. Using equation (2.6), one can find that

$$\begin{aligned}\frac{d}{d\tau}\left(G_{\mu\nu}\frac{dz^{\mu}}{d\tau}\frac{dz^{\nu}}{d\tau}\right)&=\frac{\partial G_{\mu\nu}}{\partial z^{\lambda}}\frac{dz^{\lambda}}{d\tau}\frac{dz^{\mu}}{d\tau}\frac{dz^{\nu}}{d\tau}\\&\quad+G_{\mu\nu}\frac{d^2z^{\mu}}{d\tau^2}\frac{dz^{\nu}}{d\tau}+G_{\mu\nu}\frac{dz^{\mu}}{d\tau}\frac{d^2z^{\nu}}{d\tau^2}\\&=\left(\frac{\partial G_{\rho\sigma}}{\partial z^{\lambda}}-G_{\mu\sigma}\Gamma^{\mu}_{\rho\lambda}-G_{\nu\rho}\Gamma^{\nu}_{\sigma\lambda}\right)\frac{dz^{\rho}}{d\tau}\frac{dz^{\sigma}}{d\tau}\frac{dz^{\lambda}}{d\tau}\end{aligned}$$

Taking into account equality (2.19), it is easy to see that this quantity disappears, and therefore,

$$G_{\mu\nu}\frac{dz^{\mu}}{d\tau}\frac{dz^{\nu}}{d\tau}=-C \tag{2.25}$$

where C is an integration constant determined by initial conditions. Further, since initial conditions are always chosen in such a way that $d\tau^2$ is defined by (2.11), we obtain $C=1$. Thus, equality (2.25) guarantees that formula (2.11) is used along all the particle's trajectories. Analogous initial conditions for massless particles leads to $C=0$ (where τ is changed by some other parameter σ) and the equation of motion ensures that the quantity

$$G_{\mu\nu}\frac{dz^{\nu}}{d\sigma}\frac{dz^{\mu}}{d\sigma}$$

becomes zero along all trajectories.

2.4. The Newtonian Approximation (Linearized Gravity)

In order to find the connection of our model with the Newtonian theory, consider a particle moving slowly in a weak stationary gravitational field. We proceed according to the linearized theory of gravity. Further, it is assumed that if the particle is sufficiently slow, then one can neglect $dz^i/d\tau$ ($i=1,2,3$) with respect to $dt/d\tau$, and equation (2.6) acquires the form

$$\frac{d^2 z^\mu}{d\tau^2}+\Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2=0 \tag{2.26}$$

Since the field is stationary, all time derivatives of $G_{\mu\nu}(z)$ disappear, and therefore,

$$\Gamma^\mu_{00}=-\tfrac{1}{2}G^{\mu\nu}\frac{\partial G_{00}}{\partial z^\nu}$$

Moreover, if the field is still weak, one can introduce an almost Cartesian system of coordinates in which

$$G_{\mu\nu}=\eta_{\mu\nu}+H_{\mu\nu}, \qquad |H_{\mu\nu}|\ll 1 \tag{2.27}$$

where $H_{\mu\nu}$ consists of two parts: $h_{\mu\nu}$ is due only to an external weak stationary field and $\varepsilon_{\mu\nu}+\frac{1}{4}\varepsilon^\rho_\mu\varepsilon_{\nu\rho}$ is caused by the stochastic fluctuation of the space-time metric. Thus, in the first order of $H_{\mu\nu}$, one has

$$\Gamma^\alpha_{00}=-\tfrac{1}{2}\eta^{\alpha\beta}\frac{\partial H_{00}}{\partial z^\beta}$$

Substituting this expression for the usual affine connection into the *equation of motion* (2.26), we get

$$\frac{d^2\mathbf{z}}{d\tau^2}=\frac{1}{2}\left(\frac{dt}{d\tau}\right)^2\nabla H_{00} \tag{2.28}$$

$$\frac{d^2 t}{d\tau^2}=0$$

The solution of the second equation in (2.28) is $dt/d\tau=\text{const}$, and therefore,

$$d^2\mathbf{z}/dt^2=\tfrac{1}{2}\nabla H_{00} \tag{2.29}$$

In accordance with the usual theory of gravity, the quantity h_{00} is defined by the *Newtonian potential* ϕ_N:

$$h_{00}=-2\phi_N$$

and therefore

$$H_{00}=-2\phi_N+\varepsilon_{00}(x)+\tfrac{1}{4}\varepsilon^\rho_0(x)\varepsilon_{0\rho}(x)+\tfrac{1}{2}\varepsilon^2_{00}$$

where the last term appears from the second order of $H_{\mu\nu}$. On the other hand, as is shown above, due to the stochastic properties of the space-time metric, an additional "scalar potential" (1.72),

$$\varphi_f=-\tfrac{1}{2}c^2[\varepsilon_{00}(x)+\tfrac{1}{4}\varepsilon^\rho_0(x)\varepsilon_{0\rho}(x)+\tfrac{1}{2}\varepsilon^2_{00}(x)]$$

appears in our scheme, which gives rise to the change of the Newtonian potential

$$\phi_N \to \phi'_N = \phi_N + \varphi_f$$

Thus, in a space-time with a stochastic metric the *Newtonian law* is changed and acquires the form

$$d^2\mathbf{z}/dt^2 = -\nabla\phi'_N \tag{2.30}$$

In this approximation, the space-time metric is given by the formula

$$G_{00} = -1 - 2\phi_N + \varepsilon_{00}(x) + \tfrac{1}{4}\varepsilon_0^\rho(x)\varepsilon_{0\rho}(x) + \tfrac{1}{2}\varepsilon_{00}^2(x) \tag{2.31}$$

The gravitational potential is of the order of 10^{-39} on the "surface" of the proton, 10^{-9} on the surface of the Earth, 10^{-6} for the Sun, and 10^{-4} for the white dwarf-type stars.

Finally, it should be noted that in the given case, the gravitational force acting on the particle is given by formula (1.49); there, the quantity h entering into it now takes the form $h = -G_{00}$, so that

$$h^{1/2} = 1 + \phi_N - \tfrac{1}{2}\phi_N^2 - \tfrac{1}{2}\varepsilon_{00}(1 - \phi_N) - \tfrac{1}{8}(\varepsilon_{00}^2 + \varepsilon_0^\rho\varepsilon_{0\rho})$$

and

$$\begin{aligned} -\nabla \ln\sqrt{h} = &-\nabla\phi_N + \nabla\phi_N^2 + \tfrac{1}{2}\nabla[\varepsilon_{00}(1 - 2\phi_N + 3\phi_N^2)] \\ &+ \tfrac{1}{4}\nabla[\varepsilon_{00}^2(1 - 3\phi_N + \tfrac{15}{2}\phi_N^2)] \\ &+ \tfrac{1}{8}\nabla[\varepsilon_0^\rho\varepsilon_{0\rho}(1 - 2\phi_N + 3\phi_N^2)] \end{aligned} \tag{2.32}$$

Therefore, the *potential force* is changed and its value is defined by the averaging procedure

$$\mathbf{F} = [1 + \tfrac{3}{4}D_{00,00}(0) + \tfrac{1}{4}D_{0,0\rho}^\rho(0) - 2\phi_N]\mathbf{F}_N \tag{2.33}$$

where

$$\mathbf{F}_N = -\nabla\phi_N, \qquad D_{00,00}(0) = \tfrac{5}{6}\tilde{D}(0)$$

$$D_{0,0\rho}^\rho(0) = \begin{cases} \tfrac{5}{2}\tilde{D}(0) & \text{for the Euclidean metric} \\ -\tfrac{5}{2}\tilde{D}(0) & \text{for the pseudo-Euclidean metric} \end{cases}$$

In (2.33) we have assumed $\langle\varepsilon_{00}\nabla\varepsilon_{00}\rangle_\varepsilon = \langle\varepsilon_0^\rho\nabla\varepsilon_{0\rho}\rangle_\varepsilon = 0$ for the field. When $\phi_N = 0$, i.e., the external gravitational field becomes zero, expression (2.32) coincides with the result obtained above. This is just our correspondence principle.

2.5. Change of Time Scale in Gravitational Field with Stochastic Metric

In the presence of a gravitational field the *red shift* of light frequencies should be calculated as above. In the given case, the time interval between counts is now defined by

$$\frac{dt}{\Delta\tau}=\left[-G_{\mu\nu}(z)\frac{dz^{\mu}}{dt}\frac{dz^{\nu}}{dt}\right]^{-1/2}$$

or, in particular, if the clocks are at rest, one gets

$$dt/\Delta\tau=[-G_{00}(z)]^{-1/2}$$

Hence, it follows that the ratio of frequencies for (observing at point 1) light leaving from point 2 and light coming from point 1 due to some atomic transition is given by

$$\nu_2/\nu_1=[G_{00}(z_2)/G_{00}(z_1)]^{1/2}$$

For the limiting case of a weak field

$$G_{00}=-1-2\phi_N+\varepsilon_{00}(z)+\tfrac{1}{4}\varepsilon_0^{\rho}(z)\varepsilon_{0\rho}(z)+\tfrac{1}{2}\varepsilon_{00}^2(z)$$
$$|\phi_N|,|\varepsilon|\ll 1$$

so that $\nu_2/\nu_1=1+\Delta\nu/\nu$, where

$$\begin{aligned}\Delta\nu/\nu=\langle\Delta\nu/\nu\rangle_s=\ &\phi_N(z_2)-\phi_N(z_1)\\&+\tfrac{1}{2}[\phi_N(z_2)-\phi_N(z_1)]^2-[\phi_N^2(z_2)-\phi_N^2(z_1)]\\&-\tfrac{1}{4}D_{00,00}(0)[-1-3\phi_N(z_2)+7\phi_N(z_1)]\\&-\tfrac{1}{4}D_{0,0\rho}^{\rho}(0)[\phi_N(z_1)-\phi_N(z_2)]\\&-\tfrac{1}{4}D_{00,00}(z_1-z_2)[1-\phi_N(z_2)-3\phi_N(z_1)]\end{aligned}$$

From this, we see that even in the absence of the gravitational field there exists a contribution to the red-shift value due to the stochastic fluctuation of the space-time metric; that is,

$$(\Delta\nu/\nu)_{\text{stoch}}=\tfrac{1}{4}D_{00,00}(0)-\tfrac{1}{4}D_{00,00}(z_1-z_2)$$

coinciding with formula (1.36).

3. TENSOR ANALYSIS IN SPACE-TIME WITH STOCHASTIC METRIC

3.1. Reformulation of the General Covariance Principle in the Presence of the Stochastic Metric

As is shown above, due to the stochastic or fluctuational character of space-time, the equivalence principle between gravity and inertia is achieved up to the order of the value l_{Pl}^2/l^2 or l_{Pl}^4/l^4, depending on the form of the distribution for the gravitonlike particle covariance. In the previous sections we applied the modified (or generalized) equivalence principle in order to introduce the gravitational effect into physical systems in the case of space-time with a stochastic metric. Following this, we also wrote equations in a virtual "quasilocal" inertial system of coordinates [i.e., equations of the special theory of relativity such that $d^2x^\alpha/d\tau^2=(1/m)F^\alpha$ with the stochastic metric $g_{\mu\nu}^{(s)}(x)$] and carried out a transformation of the coordinates $x^\alpha \Rightarrow z^\nu$ in order to find corresponding equations in the laboratory system of coordinates with a stochastic metric. In principle, one can use this method further; but it leads us to very tedious calculations when we arrive at the definition of field equations in electrodynamics and gravity.

Following Weinberg (1972), we here employ another method which has the same physical content, but is more elegant in its notation and more convenient to handle. This apparoach is based on the extended version of the equivalence principle known as the principle of general covariance. It asserts that a physical equation is given in an arbitrary gravitational field in the case where the following two conditions are fulfilled:

1. The equation is given in the absence of gravity, i.e., it corresponds to the laws of the special theory of relativity (in our case it is slightly modified according to Section 1) when its metric tensor $G_{\mu\nu}$ is equal to the stochastic metric $g_{\mu\nu}^{(s)}$ and the affine connection $\Gamma_{\mu\nu}^\lambda(z)$ coincides with $\gamma_{\mu\nu}^\lambda(x)$.
2. The equation is generally covariant, i.e., it preserves its form under an arbitrary transformation of coordinates $z^\nu \to z'^\nu$.

As in the usual theory of gravity, in our case the *general covariance principle* follows from the equivalence principle. When we obtain the general covariance equation, new quantities, the metric tensor $G_{\mu\nu}$ and the affine connection $\Gamma_{\mu\nu}^\lambda$, enter in. In this case, one does not need to assume that these quantities disappear totally, and that therefore any restriction on the original equation has arisen. On the contrary, we utilize the existence of $G_{\mu\nu}$ and $\Gamma_{\mu\nu}^\lambda$ in order to introduce gravitational fields.

The modified general covariance principle is used only on small scales with respect to a typical space-time size for gravitational field, since only in small domains is one guided by the equivalence principle and able to find a system of coordinates in which pure gravitational effects are absent.

3.2. Vectors and Tensors

To construct invariant physical equations with respect to the transformation of coordinates in a space-time with a stochastic metric, we must know how quantities standing in equations under this transformation behave. We start from simple physical quantities such as vectors and tensors.

By definition, as in the usual case, under the change of variables $z^{\nu} \to z'^{\nu}$ *contra-* and *covariant vectors* V^{μ} and U_{μ} transform by the formulas

$$V'^{\mu} = V^{\nu} \frac{\partial z'^{\mu}}{\partial z^{\nu}}, \qquad U'_{\mu} = \frac{\partial z^{\nu}}{\partial z'^{\mu}} U_{\nu} \tag{3.1}$$

respectively. For example, the rule of taking the partial derivative gives

$$dz'^{\mu} = \frac{\partial z'^{\mu}}{\partial z^{\nu}} dz^{\nu}$$

so that the differential of coordinates is a covariant vector. If ϕ is a scalar field, then $\partial \phi / \partial z^{\mu}$ is a covariant vector, since

$$\frac{\partial \phi}{\partial z'^{\mu}} = \frac{\partial z^{\nu}}{\partial z'^{\mu}} \frac{\partial \phi}{\partial z^{\nu}}$$

By this general rule of transformation of any physical quantity under the passage from one system of reference to another, one can easily define its value in an arbitrary system of coordinates. For example, we now define force F^{μ} in the system of references z^{μ} by knowing its value f^{α} in the local inertial system of coordinates ξ^{α}. Thus, there are three systems of reference at our disposal:

(a) The local inertial system of reference ξ^{α} with the Minkowski metric $\eta_{\alpha\beta}$.
(b) The "quasilocal" inertial system of reference x^{ν} with the stochastic metric

$$g^{(s)}_{\mu\nu}(x) = \eta_{\mu\nu} + \varepsilon_{\mu\nu}(x) + \tfrac{1}{4}\varepsilon^{\rho}_{\nu}(x)\varepsilon_{\mu\rho}(x)$$

(c) The general system of reference z^{μ} with the stochastic modified metric $G_{\mu\nu}$.

Then, the contravariant vector F^{μ}_{G} is defined by the cyclic transformations

$$F^{\mu}_{G}(z) = \frac{\partial z^{\mu}}{\partial x^{\nu}} F^{\nu}(x) = \frac{\partial z^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial \xi^{\alpha}} f^{\alpha} \tag{3.2}$$

Since the transformation matrix $\partial x^{\nu}/\partial \xi^{\alpha}$ is given by formula (1.12), we have

$$\begin{aligned} F^{\mu}_{G}(z) &= \frac{\partial z^{\mu}}{\partial x^{\nu}} [\delta^{\nu}_{\alpha} - \tfrac{1}{2}\varepsilon^{\nu}_{\alpha}(x) + \tfrac{1}{4}\varepsilon^{\rho}_{\alpha}(x)\varepsilon^{\nu}_{\rho}(x) - \cdots] f^{\alpha} \\ &= \frac{\partial z^{\mu}}{\partial z^{\nu}} f^{\nu} + \frac{\partial z^{\mu}}{\partial x^{\nu}} f^{\alpha}[-\tfrac{1}{2}\varepsilon^{\nu}_{\alpha}(x) + \tfrac{1}{4}\varepsilon^{\rho}_{\alpha}(x)\varepsilon^{\nu}_{\rho}(x) - \cdots] \end{aligned}$$

In the absence of gravity, $\partial z^{\mu}/\partial x^{\nu} = \delta^{\mu}_{\nu}$; therefore, expression (3.2) gives the previous result for the modified special theory of relativity with a stochastic metric (Section 1).

From contra- and covariant vectors one can immediately turn to high-rank tensors. For example, if $T_\nu^{\mu\lambda}$ is a *tensor* of the type of $U_\nu V^\mu W^\lambda$, then its transformation is given by

$$T_\nu'^{\mu\lambda}(z') = \frac{\partial z^\rho}{\partial z'^\nu} \frac{\partial z'^\mu}{\partial z^\varkappa} \frac{\partial z'^\lambda}{\partial z^\delta} T_\rho^{\varkappa\delta}(z)$$

A more important tensor is the *metric tensor* defined by the formula

$$g_{\mu\nu}(x) = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\nu} \frac{\partial \xi^\beta}{\partial x^\mu}$$

in an arbitrary chosen system of reference x^μ. In the general system of reference when there exists a gravitational field, the metric tensor reads

$$G_{\mu\nu}(z) = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial z^\mu} \frac{\partial \xi^\beta}{\partial z^\nu} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\rho} \frac{\partial \xi^\beta}{\partial x^\delta} \frac{\partial x^\rho}{\partial z^\mu} \frac{\partial x^\delta}{\partial z^\nu}$$

and therefore,

$$G_{\mu\nu}(z) = g_{\rho\sigma}^{(s)}(x) \frac{\partial x^\rho}{\partial z^\mu} \frac{\partial x^\sigma}{\partial z^\nu}$$

from which we see that $G_{\mu\nu}(z)$ is indeed the *covariant tensor.* An *inverse tensor* with respect to $G_{\mu\nu}(z)$ is given by the relations

$$G^{\mu\rho}(z) G_{\nu\rho}(z) = G_{\nu\rho}(z) G^{\rho\mu}(z) = \delta_\nu^\mu \tag{3.3}$$

Then

$$\begin{aligned} \frac{\partial z^\lambda}{\partial x^\rho} \frac{\partial z^\mu}{\partial x^\sigma} g_{(s)}^{\rho\sigma}(x) G_{\mu\nu}(z) &= \frac{\partial z^\lambda}{\partial x^\rho} \frac{\partial z^\mu}{\partial x^\sigma} g_{(s)}^{\rho\sigma} \frac{\partial x^\varkappa}{\partial z^\mu} \frac{\partial x^\eta}{\partial z^\nu} g_{\varkappa\eta}^{(s)} \\ &= \frac{\partial z^\lambda}{\partial x^\rho} g_{(s)}^{\rho\varkappa} \frac{\partial x^\eta}{\partial z^\nu} g_{\varkappa\eta}^{(s)} \\ &= \frac{\partial z^\lambda}{\partial x^\rho} \frac{\partial x^\rho}{\partial z^\nu} = \delta_\nu^\lambda \end{aligned}$$

and therefore, the construction

$$\frac{\partial z^\lambda}{\partial x^\rho} \frac{\partial z^\mu}{\partial x^\sigma} g_{(s)}^{\rho\sigma} = G^{\lambda\mu} \tag{3.4}$$

is just the contravariant tensor.

In accordance with the definition (2.8) in space-time with a stochastic metric the *Kronecker symbol* δ_μ^ν is a mixed tensor of the type $T_\mu^\nu = U_\mu V^\nu$, since

$$\delta_\mu^\nu \frac{\partial z^\rho}{\partial x^\mu} \frac{\partial x^\nu}{\partial z^\sigma} = \frac{\partial z^\rho}{\partial x^\mu} \frac{\partial x^\mu}{\partial z^\sigma} = \delta_\sigma^\rho \tag{3.5}$$

In addition to a scalar and zero, the Kronecker symbol δ^{μ}_{ν} (at the same time its direct products) is the unique tensor whose components are the same in any system of reference.

3.3. Tensor Algebra

As usual, in order to construct tensor equations which are invariant under arbitrary transformations of coordinates, one has to know how new tensors are formed from others. This is achieved by means of some simple algebraic operations:

1. *Summation.* The sum of tensors with the same upper and lower indices is a tensor with the same indices. Let A^{μ}_{ν} and B^{μ}_{ν} be two mixed tensors. Let us consider their sum $T^{\mu}_{\nu}=aA^{\mu}_{\nu}+bB^{\mu}_{\nu}$ for any scalar constants a and b. Then, T^{μ}_{ν} is a tensor, since

$$T'^{\mu}_{\nu}\equiv aA'^{\mu}_{\nu}+bB'^{\mu}_{\nu}=a\frac{\partial z'^{\mu}}{\partial z^{\rho}}\frac{\partial z^{\sigma}}{\partial z'^{\nu}}A^{\rho}_{\sigma}+b\frac{\partial z'^{\mu}}{\partial z^{\rho}}\frac{\partial z^{\sigma}}{\partial z'^{\nu}}B^{\rho}_{\sigma}$$

$$=\frac{\partial z'^{\mu}}{\partial z^{\rho}}\frac{\partial z^{\sigma}}{\partial z'^{\nu}}T^{\rho}_{\sigma}$$

2. *Direct Product.* The product of components of two vectors leads to a tensor, the upper and lower indices of which consist of all upper and lower indices of the two initial ones. For example, if A^{μ}_{ν} and B^{ρ} are tensors, the combination $T^{\mu\rho}_{\nu}$ is also a tensor, i.e.,

$$T'^{\mu\rho}_{\nu}\equiv A'^{\mu}_{\nu}B'^{\rho}=\frac{\partial z'^{\mu}}{\partial z^{\lambda}}\frac{\partial z^{\varkappa}}{\partial z'^{\nu}}A^{\lambda}_{\varkappa}\frac{\partial z'^{\rho}}{\partial z^{\sigma}}B^{\sigma}$$

$$=\frac{\partial z'^{\mu}}{\partial z^{\lambda}}\frac{\partial z^{\varkappa}}{\partial z'^{\nu}}\frac{\partial z'^{\rho}}{\partial z^{\sigma}}T^{\lambda\sigma}_{\varkappa}$$

3. *Contraction.* Equating the upper and lower indices and summation over their four-values gives a new tensor in which these two indices are absent. For example, if $T^{\mu\rho\sigma}_{\nu}$ is a tensor from which one can form a new quantity $T^{\mu\rho}\equiv T^{\mu\rho\nu}_{\nu}$, then $T^{\mu\rho}$ is also a tensor, since

$$T'^{\mu\rho}=T'^{\mu\rho\nu}_{\nu}=\frac{\partial z'^{\mu}}{\partial z^{\varkappa}}\frac{\partial z^{\lambda}}{\partial z'^{\nu}}\frac{\partial z'^{\rho}}{\partial z^{\eta}}\frac{\partial z'^{\nu}}{\partial z^{\tau}}T^{\varkappa\eta\tau}_{\lambda}$$

$$=\frac{\partial z'^{\mu}}{\partial z^{\varkappa}}\frac{\partial z'^{\rho}}{\partial z^{\eta}}T^{\varkappa\eta\lambda}_{\lambda}=\frac{\partial z'^{\mu}}{\partial z^{\varkappa}}\frac{\partial z'^{\rho}}{\partial z^{\eta}}T^{\varkappa\eta}$$

The above-mentioned three operations may always be united in a different way. The most important combined operation leads to *lowering and raising* *the indices*, which is achieved by means of the stochastic metric tensor $G_{\mu\nu}$. For instance, let $T^{\mu\rho}_{\sigma}$ and $S^{\rho}_{\mu\sigma}$ be tensors; then the new formations

$$S^{\rho}_{\nu\sigma} \equiv G_{\mu\nu} T^{\mu\rho}_{\sigma} \qquad \text{and} \qquad R^{\nu\rho}_{\sigma} \equiv G^{\mu\nu} S^{\rho}_{\mu\sigma}$$

are also tensors in accordance with rules 2 and 3. Owing to relations (3.3), the raising and lowering of both the indices for the *metric tensor* $G_{\mu\nu}$ are carried out by the following rules:

$$G^{\lambda\mu} G^{\varkappa\nu} G_{\nu\mu} = G^{\lambda\mu} G_{\mu\nu} G^{\nu\varkappa} = G^{\lambda\mu} \delta^{\varkappa}_{\mu} = G^{\lambda\varkappa}$$

and

$$G_{\lambda\mu} G^{\varkappa\nu} G_{\nu\mu} = G_{\lambda\mu} G_{\mu\nu} G^{\nu\varkappa} = G_{\lambda\mu} \delta^{\varkappa}_{\mu} = G_{\lambda\varkappa}$$

This rule of lowering and raising the indices for $G_{\mu\nu}$ again gives the metric tensor and its inverse, respectively.

3.4. Tensor Density

An important example of nontensor values is the *determinant of the metric tensor*

$$\hat{G} = -\text{Det}\, G_{\mu\nu}(z)$$

The rule of *metric tensor transformation* may be regarded as the matrix equation

$$G'_{\mu\nu} = \frac{\partial z^{\rho}}{\partial z'^{\mu}} G_{\rho\sigma} \frac{\partial z^{\sigma}}{\partial z'^{\nu}}$$

Calculating its *determinant*, we have

$$G' = |\partial z/\partial z'|^2 G \tag{3.6}$$

where $|\partial z/\partial z'|$ is the Jacobian of the transformation $z'^{\nu} \to z^{\nu}$, i.e., the determinant of the matrix $\partial z^{\rho}/\partial z'^{\mu}$. As in the usual case, if we do not take into account an additional multiplier caused by the Jacobian, we call a quantity of the type of G a *scalar density* in the general system of reference z^{μ} with the stochastic metric $G_{\mu\nu}$. Similarly, a value that transforms as a tensor but with additional multipliers from the Jacobian is called a *tensor density.* We call the number of factors $|\partial z'/\partial z|$ in the determinant the *weight of the density.* For example, from expression (3.6) it follows that G is a density with weight -2, since

$$|\partial z/\partial z'| = |\partial z'/\partial z|^{-1}$$

The latter is easily verified by estimating the determinant of the equation

$$\frac{\partial z^{\mu}}{\partial z'^{\lambda}} \frac{\partial z'^{\lambda}}{\partial z^{\nu}} = \delta^{\mu}_{\nu}$$

Any tensor density with weight w can be expressed as a usual tensor multiplied by the coefficient $G^{-w/2}$. For example, the tensor density F^{μ}_{ν} with weight w transforms by the rule

$$F'^{\mu}_{\nu} = \left|\frac{\partial z'}{\partial z}\right|^{w} \frac{\partial z'^{\mu}}{\partial z^{\lambda}} \frac{\partial z^{\varkappa}}{\partial z'^{\nu}} T^{\lambda}_{\varkappa}$$

Using (3.6), we find

$$G'^{w/2} F'^{\mu}_{\nu} = \frac{\partial z'^{\mu}}{\partial z^{\lambda}} \frac{\partial z^{\varkappa}}{\partial z'^{\nu}} G^{w/2} F^{\lambda}_{\varkappa}$$

An important role of tensor densities is defined by the *fundamental theorem of integral calculus*, which asserts that under an arbitrary transformation of coordinates $z^{\nu} \to z'^{\nu}$ the volume element $d^4 z$ is replaced by

$$d^4 z' = |dz'/dz|\, d^4 z \tag{3.7}$$

Therefore the product of $d^4 z$ on the tensor density with the weight -1 transforms as a usual tensor. In particular, $G^{1/2} d^4 z$ is an invariant element of the volume.

There exists an important tensor density, the components of which are the same in all systems of coordinates; that is the *Levi-Civita tensor* density,

$$\varepsilon^{\mu\nu\lambda\varkappa} = \begin{cases} +1 & \text{for even rearrangment of indices} \\ -1 & \text{for odd rearrangement of indices} \\ 0 & \text{if any pairs of indices coincide} \end{cases}$$

This quantity is the tensor density with weight -1. Multiplying $\varepsilon^{\mu\nu\lambda\varkappa}$ on $G^{-1/2}$, one can construct the usual contravariant tensor. Moreover, it is possible to form the covariant density by means of lowering its indices:

$$\varepsilon_{\rho\sigma\eta\xi} = G_{\rho\mu} G_{\sigma\nu} G_{\eta\lambda} G_{\xi\varkappa} \varepsilon^{\mu\nu\lambda\varkappa}$$

This expression is antisymmetric over indices, and therefore, it is proportional to $\varepsilon^{\rho\sigma\eta\xi}$; the coefficient of proportionality is $-G$, so that

$$\varepsilon_{\rho\sigma\eta\xi} = -G\varepsilon^{\rho\sigma\eta\xi}$$

One can easily verify that $\varepsilon_{\rho\sigma\eta\xi}$ is the covariant tensor density with weight -1.

Finally, we present a calculation method for the determinant of the metric tensor. Let the stochastic metric $G_{\mu\nu}$ be a tensor of the type

$$G_{\mu\nu} = \eta_{\mu\nu} + \varepsilon_{\mu\nu}(x) + \tfrac{1}{4}\varepsilon^{\rho}_{\mu}(x)\varepsilon_{\nu\rho}(x)$$

Then, by definition,

$$G = -\text{Det}\, G_{\mu\nu} = \text{Det}(1-\Lambda) = \prod_j (1-\lambda_j) \tag{3.8}$$

where numbers λ_i are *eigenvalues* of the matrix $\Lambda_{\mu\nu}=\varepsilon_{\mu\nu}(x)+\frac{1}{4}\varepsilon^{\rho}_{\mu}(x)\varepsilon_{\nu\rho}(x)$. After a simple transformation, expression (3.8) reads

$$\begin{aligned} G &= \exp\left[\sum_j \ln(1-\lambda_j)\right] = \exp\left[-\sum_j \sum_{n=1}^{\infty} \frac{1}{n}\lambda_j^n\right] \\ &= \exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Tr}\Lambda^n\right] \\ &= \exp[-\tfrac{1}{4}\operatorname{Tr}\varepsilon'^2]\exp\left[-\sum_{n=2}^{\infty}\frac{1}{n}\operatorname{Tr}\Lambda^n\right] \\ &= \exp[-\tfrac{1}{4}\operatorname{Tr}\varepsilon'^2-\tfrac{1}{2}\operatorname{Tr}\Lambda^2-\tfrac{1}{3}\operatorname{Tr}\Lambda^3-\cdots] \end{aligned} \tag{3.9}$$

Thus,

$$G^{1/2}\approx 1-\tfrac{1}{8}\operatorname{Tr}\varepsilon'^2-\tfrac{1}{4}\operatorname{Tr}\varepsilon^2+O(\varepsilon^3)$$

and

$$\ln\sqrt{G}=-\tfrac{1}{8}\operatorname{Tr}\varepsilon'^2-\tfrac{1}{4}\operatorname{Tr}\varepsilon^2$$

where we have used the definitions

$$\operatorname{Tr}\varepsilon'^2=\operatorname{Tr}\varepsilon^{\rho}_{\nu}(x)\varepsilon_{\mu\rho}(x)\equiv\varepsilon^{\rho}_{\nu}(x)\varepsilon_{\nu\rho}(x)$$

$$\operatorname{Tr}\varepsilon^2=\operatorname{Tr}\varepsilon_{\mu\nu}(x)\varepsilon_{\rho\sigma}(x)$$

It should be noted that the rules of the tensor algebra are easily extended to the case of tensor densities:

1. The sum of two tensor densities with the same weight w is a tensor density with weight w.

2. The direct product of two tensor densities with corresponding weights w_1 and w_2 gives a tensor density with the weight w_1+w_2.

3. The contraction of indices for a tensor density with the weight w leads to a tensor density with same weight w. From rules 2 and 3 it follows that lowering and raising the indices does not change the weight of the tensor density.

3.5. Transformation of the Affine Connection in Space-Time with Stochastic Metric

It is well known that apart from trivial tensor quantities and densities in physical laws, nontensor values may appear, among which the affine connection plays an important role in the gravitational theory. In space-time with a stochastic metric the affine connection has the same form as in the usual theory, and therefore, we follow the standard method for transformation of the affine connection. Now we separate its nonhomogeneous nontensor term. By definition,

$$\Gamma^{\lambda}_{\mu\nu}=\frac{\partial z^{\lambda}}{\partial\xi^{\alpha}}\frac{\partial^2\xi^{\alpha}}{\partial z^{\mu}\,\partial z^{\nu}}$$

where ξ^{α} is the local inertial system of coordinates. In another system of coordinates z'^{ν} the value of $\Gamma^{\lambda}_{\mu\nu}$ acquires the form

$$\begin{aligned}\Gamma'^{\lambda}_{\mu\nu}&=\frac{\partial z'^{\lambda}}{\partial\xi^{\alpha}}\frac{\partial^2\xi^{\alpha}}{\partial z'^{\mu}\,\partial z'^{\nu}}\\&=\frac{\partial z'^{\lambda}}{\partial z^{\rho}}\frac{\partial z^{\rho}}{\partial\xi^{\alpha}}\frac{\partial}{\partial z'^{\mu}}\left(\frac{\partial z^{\sigma}}{\partial z'^{\nu}}\frac{\partial\xi^{\alpha}}{\partial z^{\sigma}}\right)\\&=\frac{\partial z'^{\lambda}}{\partial z^{\rho}}\frac{\partial z^{\rho}}{\partial\xi^{\alpha}}\left(\frac{\partial z^{\sigma}}{\partial z'^{\nu}}\frac{\partial z^{\tau}}{\partial z'^{\mu}}\frac{\partial^2\xi^{\alpha}}{\partial z^{\tau}\,\partial z^{\sigma}}+\frac{\partial^2 z^{\sigma}}{\partial z'^{\mu}\,\partial z'^{\nu}}\frac{\partial\xi^{\alpha}}{\partial z^{\sigma}}\right)\end{aligned}$$

Taking into account definition (2.7), we find

$$\Gamma'^{\lambda}_{\mu\nu}=\frac{\partial z'^{\lambda}}{\partial z^{\rho}}\frac{\partial z^{\tau}}{\partial z'^{\mu}}\frac{\partial z^{\sigma}}{\partial z'^{\nu}}\Gamma^{\rho}_{\tau\sigma}+\frac{\partial z'^{\lambda}}{\partial z^{\sigma}}\frac{\partial^2 z^{\sigma}}{\partial z'^{\mu}\,\partial z'^{\nu}}\tag{3.10}$$

Here the last term makes $\Gamma'^{\lambda}_{\mu\nu}$ the exact nontensor value.

Tensor analysis permits us to establish a simple connection between $\Gamma^{\lambda}_{\mu\nu}$ and $G_{\mu\nu}$. Notice that

$$\begin{aligned}\frac{\partial}{\partial z'^{\varkappa}}G'_{\mu\nu}&=\frac{\partial}{\partial z'^{\varkappa}}\left(G_{\rho\sigma}\frac{\partial z^{\rho}}{\partial z'^{\mu}}\frac{\partial z^{\sigma}}{\partial z'^{\nu}}\right)=\frac{\partial G_{\rho\sigma}}{\partial z^{\tau}}\frac{\partial z^{\tau}}{\partial z'^{\varkappa}}\frac{\partial z^{\rho}}{\partial z'^{\mu}}\frac{\partial z^{\sigma}}{\partial z'^{\nu}}\\&\quad+G_{\rho\sigma}\frac{\partial^2 z^{\rho}}{\partial z'^{\varkappa}\,\partial z'^{\mu}}\frac{\partial z^{\sigma}}{\partial z'^{\nu}}+G_{\rho\sigma}\frac{\partial^2 z^{\rho}}{\partial z'^{\varkappa}\,\partial z'^{\nu}}\frac{\partial z^{\sigma}}{\partial z'^{\mu}}\end{aligned}$$

and therefore,

$$\begin{aligned}\frac{\partial}{\partial z'^{\mu}}G'_{\varkappa\nu}+\frac{\partial}{\partial z'^{\nu}}G'_{\varkappa\mu}-\frac{\partial}{\partial z'^{\varkappa}}G'_{\mu\nu}&=\frac{\partial z^{\tau}}{\partial z'^{\varkappa}}\frac{\partial z^{\rho}}{\partial z'^{\mu}}\frac{\partial z^{\sigma}}{\partial z'^{\nu}}\left(\frac{\partial G_{\sigma\tau}}{\partial z^{\rho}}+\frac{\partial G_{\rho\tau}}{\partial z^{\sigma}}-\frac{\partial G_{\rho\sigma}}{\partial z^{\tau}}\right)\\&\quad+2G_{\rho\sigma}\frac{\partial^2 z^{\rho}}{\partial z'^{\mu}\,\partial z'^{\nu}}\frac{\partial z^{\sigma}}{\partial z'^{\varkappa}}\end{aligned}$$

From which it follows that

$$\left\{\begin{matrix}\lambda\\ \mu\nu\end{matrix}\right\}'=\frac{\partial z'^{\lambda}}{\partial z^{\rho}}\frac{\partial z^{\tau}}{\partial z'^{\mu}}\frac{\partial z^{\sigma}}{\partial z'^{\nu}}\left\{\begin{matrix}\rho\\ \tau\sigma\end{matrix}\right\}+\frac{\partial z'^{\lambda}}{\partial z^{\rho}}\frac{\partial^2 z^{\rho}}{\partial z'^{\mu}\,\partial z'^{\nu}}\tag{3.11}$$

where

$$\left\{\begin{matrix}\lambda\\ \mu\nu\end{matrix}\right\}\equiv\tfrac{1}{2}G^{\lambda\varkappa}\left[\frac{\partial G_{\varkappa\nu}}{\partial z^{\mu}}+\frac{\partial G_{\varkappa\mu}}{\partial z^{\nu}}-\frac{\partial G_{\mu\nu}}{\partial z^{\varkappa}}\right]\tag{3.12}$$

Subtracting (3.11) from (3.10), we see that quantity $\Gamma^{\lambda}_{\mu\nu}-\{{}^{\lambda}_{\mu\nu}\}$ is a tensor, since

$$\left[\Gamma^{\lambda}_{\mu\nu}-\begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix}\right]'=\frac{\partial z'^{\lambda}}{\partial z^{\rho}}\frac{\partial z^{\tau}}{\partial z'^{\mu}}\frac{\partial z^{\sigma}}{\partial z'^{\nu}}\left[\Gamma^{\rho}_{\tau\sigma}-\begin{Bmatrix}\rho\\ \tau\sigma\end{Bmatrix}\right] \tag{3.13}$$

The modified equivalence principle of the second level tells us that there exists a quasilocal-inertial system of reference in which effects of external gravitational fields are absent. According to the *correspondence principle* employing the equivalence principle of the first level, in this system of reference when an external gravitational field disappears, the affine connection $\Gamma^{\lambda}_{\mu\nu}$ and the stochastic metric $G_{\mu\nu}$ coincide with $\gamma^{\lambda}_{\mu\nu}$ and $g^{(s)}_{\mu\nu}$, respectively. Since by definition (2.24) in this system of reference the expression

$$\left[\Gamma^{\rho}_{\tau\mu}-\begin{Bmatrix}\rho\\ \tau\mu\end{Bmatrix}\right]\Rightarrow\left[\gamma^{\rho}_{\tau\mu}-\begin{Bmatrix}\rho\\ \tau\mu\end{Bmatrix}\right]\Rightarrow 0$$

becomes zero and at the same time is a tensor value, so that it should disappear in any arbitrary chosen system of reference, and therefore,

$$\Gamma^{\lambda}_{\mu\nu}=\begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix}$$

Now we give another expression for the nonhomogeneous term in the transformation rule of $\Gamma^{\lambda}_{\mu\nu}$. Differentiate the identity

$$\frac{\partial z'^{\lambda}}{\partial z^{\rho}}\frac{\partial z^{\rho}}{\partial z'^{\nu}}=\delta^{\lambda}_{\nu}$$

with respect to z'^{μ}, from which it follows immediately that

$$\frac{\partial z'^{\lambda}}{\partial z^{\rho}}\frac{\partial^2 z^{\rho}}{\partial z'^{\nu}\,\partial z'^{\mu}}=-\frac{\partial z^{\rho}}{\partial z'^{\nu}}\frac{\partial z^{\sigma}}{\partial z'^{\mu}}\frac{\partial^2 z'^{\lambda}}{\partial z^{\rho}\,\partial z^{\sigma}} \tag{3.14}$$

Therefore, expression (3.10) may be written as

$$\Gamma'^{\lambda}_{\mu\nu}=\frac{\partial z'^{\lambda}}{\partial z^{\rho}}\frac{\partial z^{\tau}}{\partial z'^{\mu}}\frac{\partial z^{\sigma}}{\partial z'^{\nu}}\Gamma^{\rho}_{\tau\sigma}-\frac{\partial z^{\rho}}{\partial z'^{\nu}}\frac{\partial z^{\sigma}}{\partial z'^{\mu}}\frac{\partial^2 z'^{\lambda}}{\partial z^{\rho}\,\partial z^{\sigma}} \tag{3.15}$$

This is just the expression which would be obtained by carrying out the inverse transformation $z'^{\nu}\Rightarrow z^{\nu}$ and solving the obtained equality with respect to $\Gamma'^{\lambda}_{\mu\nu}$.

Now we are able to use the general covariance principle in order to prove that an "almost freely" falling particle satisfies the following *equation of motion*:

$$\frac{d^2 z^{\mu}}{d\tau^2}+\Gamma^{\mu}_{\nu\lambda}(z)\frac{dz^{\nu}}{d\tau}\frac{dz^{\lambda}}{d\tau}=0 \tag{3.16}$$

where the proper time $d\tau^2$ is given by formula (2.11). First notice that equations (3.16) and (2.11) are valid in the absence of gravity, since

$$\Gamma^{\mu}_{\nu\lambda}(z) \Rightarrow \gamma^{\mu}_{\nu\lambda}(x) \qquad \text{and} \qquad G_{\mu\nu}(z) \Rightarrow g^{(s)}_{\mu\nu}(x)$$
$$\frac{d^2x^\mu}{d\tau^2} + \gamma^{\mu}_{\nu\lambda}(x)\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \tag{3.17}$$

But this coincides with the equations that describe a "free" particle in the special theory of relativity modified in accordance with our assumption. Further, notice that (3.16) and (2.11) are invariant under an arbitrary transformation of coordinates, since

$$\frac{d^2z'^\mu}{d\tau^2} = \frac{d}{d\tau}\left(\frac{\partial z'^\mu}{\partial z^\nu}\frac{dz^\nu}{d\tau}\right) = \frac{\partial z'^\mu}{\partial z^\nu}\frac{d^2z^\nu}{d\tau^2} + \frac{\partial^2 z'^\mu}{\partial z^\nu\,\partial z^\lambda}\frac{dz^\lambda}{d\tau}\frac{dz^\nu}{d\tau}$$

whereas relation (3.15) leads to

$$\Gamma'^{\mu}_{\sigma\delta}\frac{dz'^\sigma}{d\tau}\frac{dz'^\delta}{d\tau} = \frac{\partial z'^\mu}{\partial z^\nu}\Gamma^{\nu}_{\lambda\rho}\frac{dz^\lambda}{d\tau}\frac{dz^\rho}{d\tau} - \frac{\partial^2 z'^\mu}{\partial z^\nu\,\partial z^\lambda}\frac{dz^\lambda}{d\tau}\frac{dz^\nu}{d\tau}$$

Adding these two equations, we find that the left part of equation (3.16) is a vector, i.e.,

$$\frac{d^2z'^\mu}{d\tau^2} + \Gamma'^{\mu}_{\nu\lambda}\frac{dz'^\nu}{d\tau}\frac{dz'^\lambda}{d\tau} = \frac{\partial z'^\mu}{\partial z^\delta}\left(\frac{d^2z^\delta}{d\tau^2} + \Gamma^{\delta}_{\sigma\rho}\frac{dz^\sigma}{d\tau}\frac{dz^\rho}{d\tau}\right) \tag{3.18}$$

Thus, equations (3.16) and (2.11) turn out to be exactly covariant in space-time with the stochastic metric $G_{\mu\nu}$. The general covariance principle of the second level tells us that relations (3.16) and (2.11) are valid in arbitrary gravitational fields, since they are indeed satisfied in quasilocal inertial system of references. Moreover, we recall the analogous situation which asserts that relations are valid in all systems of reference (including those with stochastic metric) if they are valid in any system.

3.6. Covariant Differentiation

As in the usual theory of tensor analysis, we can easily generalize the definition of covariant differentiation in space-time with a stochastic metric. Generally speaking, differentiation of a tensor does not lead to a new tensor. Now we turn to the definition of covariant differentiation by using the affine connection $\Gamma^{\lambda}_{\mu\nu}$. In this connection it should be noted that, using $\gamma^{\lambda}_{\mu\nu}$ for an additional fictitious "gravitational" field, one may also formulate covariant differentiation with respect to variables x^ν with stochastic metric $g^{(s)}_{\mu\nu}$. Thus, consider the contravariant vector V^μ, the transformation rule of which is

$$V'^\mu = \frac{\partial z'^\mu}{\partial z^\nu} V^\nu$$

Differentiation of this equality with respect to z'^λ gives

$$\frac{\partial V'^\mu}{\partial z'^\lambda} = \frac{\partial z'^\mu}{\partial z^\nu} \frac{\partial z^\rho}{\partial z'^\lambda} \frac{\partial V^\nu}{\partial z^\rho} + \frac{\partial^2 z'^\mu}{\partial z^\nu \partial z^\rho} \frac{\partial z^\rho}{\partial z'^\lambda} V^\nu \tag{3.19}$$

The first term on the right-hand side of this equation coincides with what would have arisen if the expression $\partial V^\mu/\partial z^\lambda$ were a tensor, but the second term breaks the tensor character $\partial V'^\mu/\partial z'^\lambda$. Although $\partial V^\mu/\partial z^\lambda$ is not a tensor, by means of it one can construct a tensor. Using equation (3.15), we find

$$\begin{aligned} \Gamma'^\mu_{\lambda\varkappa} V'^\varkappa &= \left(\frac{\partial z'^\mu}{\partial z^\nu} \frac{\partial z^\rho}{\partial z'^\lambda} \frac{\partial z^\sigma}{\partial z'^\varkappa} \Gamma^\nu_{\rho\sigma} - \frac{\partial^2 z'^\mu}{\partial z^\rho \partial z^\sigma} \frac{\partial z^\rho}{\partial z'^\lambda} \frac{\partial z^\sigma}{\partial z'^\varkappa} \right) \frac{\partial z'^\varkappa}{\partial z^\delta} V^\delta \\ &= \frac{\partial z'^\mu}{\partial z^\nu} \frac{\partial z^\rho}{\partial z'^\lambda} \Gamma^\nu_{\rho\sigma} V^\sigma - \frac{\partial^2 z'^\mu}{\partial z^\rho \partial z^\sigma} \frac{\partial z^\rho}{\partial z'^\lambda} V^\sigma \end{aligned} \tag{3.20}$$

Adding (3.19) to (3.20), we see that nonhomogeneous terms cancel each other and the result reads

$$\frac{\partial V'^\mu}{\partial z'^\lambda} + \Gamma'^\mu_{\lambda\varkappa} V'^\varkappa = \frac{\partial z'^\mu}{\partial z^\nu} \frac{\partial z^\rho}{\partial z'^\lambda} \left(\frac{\partial V^\nu}{\partial z^\rho} + \Gamma^\nu_{\rho\delta} V^\delta \right) \tag{3.21}$$

Thus, we arrive at the definition of the *covariant derivative* in space-time with a stochastic metric

$$V^\mu_{;\lambda} \equiv \frac{\partial V^\mu}{\partial z^\lambda} + \Gamma^\mu_{\lambda\delta} V^\delta \tag{3.22}$$

and equation (3.21) tells us that $V^\mu_{;\lambda}$ is a tensor, since

$$V'^\mu_{;\lambda} = \frac{\partial z'^\mu}{\partial z^\nu} \frac{\partial z^\rho}{\partial z'^\lambda} V^\nu_{;\rho} \tag{3.23}$$

We can also define the covariant derivative of a covariant vector U_μ. Recall the rule of transformation

$$U'_\mu = \frac{\partial z^\rho}{\partial z'^\mu} U_\rho$$

Differentiating this relation with respect to z'^ν, we get

$$\frac{\partial U'_\mu}{\partial z'^\nu} = \frac{\partial z^\rho}{\partial z'^\mu} \frac{\partial z^\sigma}{\partial z'^\nu} \frac{\partial U_\rho}{\partial z^\sigma} + \frac{\partial^2 z^\rho}{\partial z'^\mu \partial z'^\nu} U_\rho \tag{3.24}$$

Further, from (3.10) it follows that

$$\Gamma'^{\lambda}_{\mu\nu} U'_{\lambda} = \left(\frac{\partial z'^{\lambda}}{\partial z^{\tau}} \frac{\partial z^{\rho}}{\partial z'^{\mu}} \frac{\partial z^{\sigma}}{\partial z'^{\nu}} \Gamma^{\tau}_{\rho\sigma} + \frac{\partial z'^{\lambda}}{\partial z^{\tau}} \frac{\partial^2 z^{\tau}}{\partial z'^{\mu} \partial z'^{\nu}} \right) \frac{\partial z^{\varkappa}}{\partial z'^{\lambda}} U_{\varkappa}$$

$$= \frac{\partial z^{\rho}}{\partial z'^{\mu}} \frac{\partial z^{\sigma}}{\partial z'^{\nu}} \Gamma^{\varkappa}_{\rho\sigma} U_{\varkappa} + \frac{\partial^2 z^{\rho}}{\partial z'^{\mu} \partial z'^{\nu}} U_{\rho} \tag{3.25}$$

By subtracting (3.25) from (3.24), the nonhomogeneous terms cancel and we obtain

$$\frac{\partial U'_{\mu}}{\partial z'^{\nu}} - \Gamma'^{\lambda}_{\mu\nu} U'_{\lambda} = \frac{\partial z^{\rho}}{\partial z'^{\mu}} \frac{\partial z^{\sigma}}{\partial z'^{\nu}} \left(\frac{\partial U_{\rho}}{\partial z^{\sigma}} - \Gamma^{\varkappa}_{\rho\sigma} U_{\varkappa} \right) \tag{3.26}$$

Thus, we are able to introduce a definition of the *covariant derivative* of the covariant vector

$$U_{\mu;\nu} = \frac{\partial U_{\mu}}{\partial z^{\nu}} - \Gamma^{\lambda}_{\mu\nu} U_{\lambda} \tag{3.27}$$

and expression (3.26) tells us that $U'_{\mu;\nu}$ is a tensor, since

$$U'_{\mu;\nu} = \frac{\partial z^{\rho}}{\partial z'^{\mu}} \frac{\partial z^{\sigma}}{\partial z'^{\nu}} U_{\rho;\sigma}$$

Extension of the given method to the case of a general form of tensors encounters no difficulty. For example, let $T^{\mu\sigma}_{\lambda}$ be a tensor of the type $V^{\mu} W^{\sigma} U_{\lambda}$; then its covariant derivative is given by the standard version:

$$T^{\mu\sigma}_{\lambda;\rho} = \frac{\partial}{\partial z^{\rho}} T^{\mu\sigma}_{\lambda} + \Gamma^{\mu}_{\rho\nu} T^{\nu\sigma}_{\lambda} + \Gamma^{\sigma}_{\rho\nu} T^{\mu\nu}_{\lambda} - \Gamma^{\varkappa}_{\lambda\rho} T^{\mu\sigma}_{\varkappa} \tag{3.28}$$

where $\Gamma^{\mu}_{\rho\nu}$ is constructed by means of the stochastic metric $G_{\mu\nu}$ and it is easy to verify that expression (3.28) is indeed a tensor. Moreover, combination of covariant differentiation with the algebraic operations defined in Section 3.3 leads to the analogous rule of the usual differentiation [for details, see Weinberg (1972)].

Notice that the *covariant derivative of the stochastic metric* tensor is equal to zero for any system of reference. Indeed, by using a definition of the type (3.28), we get

$$G_{\mu\nu;\lambda} = \frac{\partial G_{\mu\nu}}{\partial z^{\lambda}} - \Gamma^{\rho}_{\lambda\mu} G_{\rho\nu} - \Gamma^{\rho}_{\lambda\nu} G_{\rho\mu}$$

Further, from equation (2.19) it follows that this quantity disappears:

$$G_{\mu\nu;\lambda} = 0$$

in space-time with a stochastic metric. In accordance with our construction (Section 1), it turns out also to be zero in the quasilocal inertial system of reference, when $\Gamma^{\lambda}_{\mu\nu} \Rightarrow \gamma^{\lambda}_{\mu\nu}$ and $G_{\mu\nu} \Rightarrow g^{(s)}_{\mu\nu}$ and the tensor is equal to zero in one system of reference; it also becomes zero in all systems of reference, including those with a stochastic metric.

3.7. Covariant Differentiation along the Curve

Up to now we have considered tensor fields defined on the whole of space-time. Here we consider a tensor $T(\tau)$ given along the curve $Z^{\nu}(\tau)$. Such types of tensors are the momentum $P^{\mu}(\tau)$ and the spin $S_{\mu}(\tau)$ of an individual particle. Of course, for such tensors it is not possible to talk about covariant differentiation over z^{ν}, but we can define the covariant derivative over the invariant quantity τ by means of which the curve is parametrized.

Let us consider the contravariant vector $A^{\nu}(\tau)$ transforming by the rule

$$A'^{\mu}(\tau) = \frac{\partial z'^{\mu}}{\partial z^{\nu}} A^{\nu}(\tau) \tag{3.29}$$

where the partial derivative $\partial z'^{\mu}/\partial z^{\nu}$ is calculated at $Z^{\nu} = Z^{\nu}(\tau)$, so that it depends on τ. Differentiating (3.29) over τ, we obtain two terms

$$\frac{dA'^{\mu}(\tau)}{d\tau} = \frac{\partial z'^{\mu}}{dz^{\nu}} \frac{dA^{\nu}(\tau)}{d\tau} + \frac{dz^{\lambda}}{d\tau} \frac{\partial^2 z'^{\mu}}{\partial z^{\nu}\, \partial z^{\lambda}} A^{\nu}(\tau) \tag{3.30}$$

The second derivatives $\partial^2 z'^{\mu}/\partial z^{\nu}\, \partial z^{\lambda}$ are similar to the term that breaks the homogeneity of the transformation rule (3.15) for the affine connection, so that we can define the covariant derivative along the curve $Z^{\nu}(\tau)$ as follows:

$$\frac{DA^{\mu}}{D\tau} \equiv \frac{dA^{\mu}}{d\tau} + \Gamma^{\mu}_{\nu\lambda} \frac{dz^{\lambda}}{d\tau} A^{\nu} \tag{3.31}$$

Then expressions (3.15), (3.29), and (3.30) show that this quantity is a vector, since

$$\frac{DA'^{\mu}}{D\tau} = \frac{\partial z'^{\mu}}{\partial z^{\nu}} \frac{DA^{\nu}}{D\tau} \tag{3.32}$$

The similarity of formulas (3.31) and (3.22) for the covariant derivative of the vector field is obvious.

Analogous considerations allow us to introduce the covariant derivative along curve $Z^{\mu}(\tau)$ for the covariant vector $U_{\mu}(\tau)$:

$$\frac{DU_{\mu}}{D\tau} = \frac{dU_{\mu}}{d\tau} - \Gamma^{\lambda}_{\mu\nu} \frac{dz^{\nu}}{d\tau} U_{\lambda} \tag{3.33}$$

Expression (3.10) permits us to verify easily that the obtained value is indeed a vector,

$$\frac{DU'_\mu}{D\tau}=\frac{\partial z^\nu}{\partial z'^\mu}\frac{DU_\nu}{D\tau} \tag{3.34}$$

Notice that all properties of the covariant differentiation expounded in Section 3.6 can be easily extended to the case of differentiation along the curve. One can consider the case when the vector $A^\mu(\tau)$ transferring along the curve (trajectory) of the particle does not change with the "time" variable τ if the particle is considered within the system of reference $x^\nu(\tau)$, i.e., in the quasilocal inertial system of reference with stochastic metric $g^{(s)}_{\mu\nu}$. As seen in Section 1, in this system of reference x^ν

$$\frac{DA^\mu}{D\tau}=0$$

This assertion is valid in all systems of references in accordance with the covariant character of differentiation along the curve $x^\nu(\tau)$. Then the vector A^μ satisfies the first-order differential equation

$$\frac{dA^\mu}{d\tau}=-\Gamma^\mu_{\nu\lambda}\frac{dx^\lambda}{d\tau}A^\nu \tag{3.35}$$

which defines vectors A^μ for all τ if A^μ is defined at some initial value of τ. In this case, it says that vector $A^\mu(\tau)$ on the curve $x^\nu(\tau)$ is defined by means of *parallel translation.* Thus, one can define any tensor on the curve $x^\nu(\tau)$, provided that its covariant derivative along this curve has disappeared.

3.8. Gradient, Curl, and Divergence in Space-Time with Stochastic Metric

Here we consider some consequences of the definition of covariant differentiation in space-time with a stochastic metric. In this case, no essential difference in the calculation of gradient, curl, and divergence appears with respect to the usual theory of the tensor analysis. There exist particular cases when the covariant derivative has a very simple form. For example, the covariant derivative of a scalar quantity coincides with the usual *gradient*:

$$T_{;\mu}=\partial T/\partial z^\mu \tag{3.36}$$

Another simple particular case is the *covariant curl.* Recalling the definition

$$U_{\mu;\nu}\equiv\partial U_\mu/\partial z^\nu-\Gamma^\lambda_{\mu\nu}U_\lambda$$

and taking into account the fact that $\Gamma^{\lambda}_{\mu\nu}$ is symmetric over indices μ and ν, one can easily see that the *covariant curl* coincides with the usual one,

$$U_{\mu;\nu} - U_{\nu;\mu} = \partial U_\mu/\partial z^\nu - \partial U_\nu/\partial z^\mu \tag{3.37}$$

For completeness, we consider the *covariant divergence* of the contravariant vector

$$V^\mu_{;\mu} \equiv \partial V^\mu/\partial z^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda \tag{3.38}$$

Notice that

$$\Gamma^\mu_{\mu\lambda} = \tfrac{1}{2} G^{\mu\rho}\left(\frac{\partial G_{\rho\mu}}{\partial z^\lambda} + \frac{\partial G_{\rho\lambda}}{\partial z^\mu} - \frac{\partial G_{\mu\lambda}}{\partial z^\rho}\right) = \tfrac{1}{2} G^{\mu\rho}\frac{\partial G_{\rho\mu}}{\partial z^\lambda} \tag{3.39}$$

This is easy to calculate if we use the definition

$$\mathrm{Tr}\left\{M^{-1}(z)\frac{\partial}{\partial z^\lambda} M(z)\right\} = \frac{\partial}{\partial z^\lambda} \ln \mathrm{Det}\, M(z) \tag{3.40}$$

for an arbitrary matrix M, where by Det we take the *determinant* and by Tr we take the *trace*, i.e., the sum of diagonal elements. Following Weinberg (1972), to prove (3.40), consider the variation of Det M with respect to the displacement of coordinates z^λ by the value δz^λ:

$$\begin{aligned}\delta \ln \mathrm{Det}\, M &\equiv \ln \mathrm{Det}(M+\delta M) - \ln \mathrm{Det}\, M\\ &= \ln[\mathrm{Det}(M+\delta M)/\mathrm{Det}\, M]\\ &= \ln \mathrm{Det}\, M^{-1}(M+\delta M)\\ &= \ln \mathrm{Det}[1 + M^{-1}\,\delta M]\\ &\Rightarrow \ln[1+\mathrm{Tr}\, M^{-1}\,\delta M] \to \mathrm{Tr}\, M^{-1}\,\delta M\end{aligned}$$

Insering the coefficient δz^λ into both sides of this expression, we see that the relation (3.40) holds. Making use of (3.40) for the case when the matrix M is equal to $G_{\rho\mu}$ and taking into account (3.39), we find

$$\Gamma^\mu_{\mu\lambda} = \tfrac{1}{2}\,\partial \ln G/\partial z^\lambda = G^{-1/2}\,\partial(G)^{1/2}/\partial z^\lambda \tag{3.41}$$

From (3.38) it follows that the covariant derivative is

$$V^\mu_{;\mu} = G^{-1/2}\,\partial(G^{1/2} V^\mu)/\partial z^\mu \tag{3.42}$$

a direct consequence of which is the covariant form of *Gauss theorem*: if V^μ becomes zero at infinity, then

$$\int d^4z\, G^{1/2} V^\mu_{;\mu} = 0 \tag{3.43}$$

Notice that due to the appearance of the coefficient $G^{1/2}$ in (3.43), the volume element $d^4z(G)^{1/2}$ is invariant.

One can also use (3.41) for the simplification of a formula for the covariant derivative of a tensor quantity. For example, by definition,

$$T^{\mu\nu}_{;\mu} \equiv \partial T^{\mu\nu}/\partial z^{\mu} + \Gamma^{\mu}_{\mu\lambda} T^{\lambda\nu} + \Gamma^{\nu}_{\mu\lambda} T^{\mu\lambda}$$

and using (3.41), we find

$$T^{\mu\nu}_{;\mu} = G^{-1/2}\,\partial(G^{1/2} T^{\mu\nu})/\partial z^{\mu} + \Gamma^{\nu}_{\mu\lambda} T^{\mu\lambda} \qquad (3.44)$$

In the particular case $T^{\mu\lambda} = -T^{\lambda\mu}$ the last term disappears, and therefore,

$$A^{\mu\nu}_{;\mu} = G^{-1/2}\,\partial(G^{1/2} A^{\mu\nu})/\partial z^{\mu} \qquad (3.45)$$

where $A^{\mu\nu}$ is an antisymmetric tensor. Moreover, the important formula

$$A_{\mu\nu;\lambda} + A_{\lambda\mu;\nu} + A_{\nu\lambda;\mu} = \frac{\partial A_{\mu\nu}}{\partial z^{\lambda}} + \frac{\partial A_{\lambda\mu}}{\partial z^{\nu}} + \frac{\partial A_{\nu\lambda}}{\partial z^{\mu}} \qquad (3.46)$$

may be obtained for the covariant differentiation of the antisymmetric covariant tensor $A_{\mu\nu} = -A_{\nu\mu}$ in space-time with a stochastic metric.

4. INFLUENCE OF GRAVITY WITH STOCHASTIC METRIC ON PHYSICAL PROCESSES

In previous sections we presented a concrete method of introducing fictitious (or background radiation) and true gravitational fields into physical systems from the point of view of a stochastic metric. Here, we study the influence of both these fields on the physical processes and explain their general and specific properties in the framework of the general covariance principle. To obtain equations of mechanics and electrodynamics in the presence of arbitrary gravitational fields, we must first write these equations in the special theory of relativity, and then explain how any quantity entering into these equations is changed under arbitrary transformations of coordinates, and replace:

(a) $\eta_{\mu\nu} \to g^{(s)}_{\mu\nu}$ for a background radiation field $\varepsilon_{\mu\nu}(x)$

(b) $g^{(s)}_{\mu\nu} \to G_{\mu\nu}$ for an external gravitational field with stochastic metric

and all derivatives by covariant ones.

For example, for a vector field A^{ν}, the corresponding formulas take the form

(c) $$\frac{dA^{\lambda}}{d\tau} \Rightarrow \frac{D^{(1)}A^{\lambda}}{D^{(1)}\tau} \equiv \frac{dA^{\lambda}}{d\tau} + \gamma^{\lambda}_{\mu\nu} \frac{dx^{\mu}}{d\tau} A^{\nu}$$

for the background radiation field $\varepsilon_{\mu\nu}(x)$; and

(d) $$\frac{dA^{\lambda}}{d\tau} \Rightarrow \frac{DA^{\lambda}}{D\tau} = \frac{dA^{\lambda}}{d\tau} + \Gamma^{\lambda}_{\mu\nu} \frac{dz^{\mu}}{d\tau} A^{\nu}$$

for an arbitrary gravitational field with stochastic metric.

Equations obtained in this way will be generally covariant and justified in the absence of gravity, and therefore they are valid in arbitrary gravitational fields provided that the given system is sufficiently small with respect to the scales of the fields.

4.1. Mechanics of a Particle

Let us consider a mechanical system in the special theory of relativity. When external fields are absent, a particle possesses permanent four-velocity U^α and constant spin value S_α, i.e., in the inertial system of reference ξ^α

$$dU^\alpha/d\tau=0 \qquad (U^\alpha \equiv d\xi^\alpha/d\tau) \tag{4.1}$$

$$dS_\alpha/d\tau=0, \qquad d\tau^2=-\eta_{\alpha\beta}\, d\xi^\alpha\, d\xi^\beta \tag{4.2}$$

Recall that spin S_α is defined in the rest system of the particle, where its value is $S_\alpha=\{\mathrm{S}, 0\}$, so that in an arbitrary Lorentz system of reference the condition

$$S_\alpha U^\alpha=0 \tag{4.3}$$

is fulfilled.

Further, according to the prescriptions of the modified general covariance principle, we must write these equations in an arbitrarily chosen system of reference z^μ by means of covariant derivatives $DU^\mu/D\tau$ and $DS_\mu/D\tau$, which become the usual ones when $\Gamma^\mu_{\nu\lambda}=0$. Thus, the correct equations giving the position and spin of the particle in an arbitrary system of reference z^ν are

$$DU^\mu/D\tau=0, \qquad DS_\mu/D\tau=0 \tag{4.4}$$

or, in more detailed form,

$$\begin{aligned} dU^\mu/d\tau+\Gamma^\mu_{\nu\lambda}U^\nu U^\lambda &=0 \\ dS_\mu/d\tau-\Gamma^\lambda_{\mu\nu}U^\nu S_\lambda &=0 \end{aligned} \tag{4.5}$$

Moreover, equality (4.3) should be written as

$$S_\mu U^\mu=0 \tag{4.6}$$

In expressions (4.5) and (4.6) the vectors U^μ and S_μ are given by

$$\begin{aligned} U^\mu &\equiv (\partial z^\mu/\partial\xi^\alpha)U^\alpha_f=\partial z^\mu/\partial\tau \\ S_\mu &\equiv (\partial\xi^\alpha/\partial z^\mu)S_{f\alpha} \end{aligned} \tag{4.7}$$

where U^α_f and $S_{f\alpha}$ are the components of U^μ and S_μ in the freely falling coordinate system ξ^α. In particular, in the presence of a fictitious radiation stochastic field $\varepsilon_{\mu\nu}(x)$ only, equations (4.5) and definitions (4.7) take the form

$$du^\mu/d\tau + \gamma^\mu_{\nu\lambda} u^\nu u^\lambda = 0$$
$$ds_\mu/d\tau - \gamma^\lambda_{\mu\nu} u^\nu s_\lambda = 0 \tag{4.8}$$

and

$$u^\mu = (\partial x^\mu/\partial \xi^\alpha) U^\alpha_f, \qquad s_\mu = (\partial \xi^\alpha/\partial x^\mu) S_{f\alpha} \tag{4.9}$$

Here the Jacobian of transformations $\partial \xi^\alpha/\partial x^\mu$, $\partial x^\nu/\partial \xi^\alpha$, and the "affine" connection $\gamma^\mu_{\nu\lambda}$ are given by formulas (1.10), (1.12), and (1.6a) [or (1.6b)], respectively. In Section 1 we considered some consequences of the first equation in (4.8). Notice that according to the general covariance principle discussed in Section 3.1, equations (4.5) are valid in the presence of gravitational fields, since they are general covariants and are valid in the absence of gravity, i.e., equations (4.5) become equations (4.1) and (4.2) when $\Gamma^\lambda_{\mu\nu}$ disappears. Thus, we see that in space-time with a stochastic metric, the equation of motion and spin of the particle are determined by the same form of equations as in the usual theory of gravity.

When an external force exists, then the covariant differentiation $DU^\nu/D\tau$ is not equal to zero, and instead of the first equation in (4.5) it is necessary to write

$$DU^\mu/D\tau = (1/m) f^\mu \tag{4.10}$$

where m is the mass of the particle and f^μ is a contravariant vector of force which may be written in an arbitrarily chosen system of reference z^μ:

$$f^\mu = (dz^\mu/\partial \xi^\alpha) f^\alpha_f$$

by using its value f^α_f in the freely falling system of reference ξ^α. One can write equation (4.10) in the usual form

$$m\, d^2 z^\mu/d\tau^2 = f^\mu - m\Gamma^\mu_{\nu\lambda}(dz^\nu/d\tau)(dz^\lambda/d\tau) \tag{4.11}$$

The term containing $\Gamma^\mu_{\nu\lambda}$ plays a *gauge potential role* in the presence of the stochastic metric $G_{\mu\nu}(z)$.

It should be noted that, in accordance with the correspondence principle, when an external gravitational force disappears, then equation (4.11) becomes

$$m\, d^2 x^\mu/d\tau^2 = F^\mu - m\gamma^\mu_{\nu\lambda}(dx^\nu/d\tau)(dx^\lambda/d\tau) \tag{4.12}$$

for the gravitational vacuumlike fictitious radiation field $\varepsilon_{\mu\nu}(x)$, where

$$F^\mu(x) = (\partial x^\mu/\partial \xi^\rho) f^\rho_f \tag{4.13}$$

is an external force of nongravitational origin. For the cases of (4.12) and (4.13), the stochastic metric is given by the formula (1.11).

4.2. Electrodynamics in Space-Time with Stochastic Metric

Recall that in the absence of gravitational and background additional fictitious fields *Maxwell's equations* in electrodynamics transcribe as

$$\partial \tilde{F}^{\alpha\beta}/\partial \xi^{\alpha} = -\tilde{J}^{\beta} \tag{4.14}$$

$$\partial \tilde{F}_{\beta\gamma}/\partial \xi^{\alpha} + \partial \tilde{F}_{\gamma\alpha}/\partial \xi^{\beta} + \partial \tilde{F}_{\alpha\beta}/\partial \xi^{\gamma} = 0 \tag{4.15}$$

where $\tilde{J}^{\beta}$ is the four-vector $\{\mathbf{J}, \tilde{\rho}\}$ and $\tilde{F}^{\alpha\beta}$ is the *tensor of the electromagnetic field* defined by the formula

$$\tilde{F}^{\alpha\beta} = \begin{pmatrix} 0 & -\tilde{E}_1 & -\tilde{E}_2 & -\tilde{E}_3 \\ \tilde{E}_1 & 0 & \tilde{B}_3 & -\tilde{B}_2 \\ \tilde{E}_2 & -\tilde{B}_3 & 0 & \tilde{B}_1 \\ \tilde{E}_3 & \tilde{B}_2 & -\tilde{B}_1 & 0 \end{pmatrix}$$

Assume that we define $F^{\mu\nu}$ and J^{μ} in arbitrary coordinates, providing that they lead to $\tilde{F}^{\alpha\beta}$ and $\tilde{J}^{\beta}$ in the local-inertial system of coordinates, and behave as tensors under arbitrary transformations of coordinates, i.e., if $\tilde{F}^{\alpha\beta}$ and $\tilde{J}^{\alpha}$ are quantities measured in the local-inertial system of reference ξ^{α}, then the relations

$$F^{\mu\nu} \equiv \frac{\partial z^{\mu}}{\partial \xi^{\alpha}} \frac{\partial z^{\nu}}{\partial \xi^{\beta}} \tilde{F}^{\alpha\beta} \qquad \text{and} \qquad J^{\mu} \equiv \frac{\partial z^{\mu}}{\partial \xi^{\alpha}} \tilde{J}^{\alpha}$$

are valid in any system of reference z^{ν}. Thus, one can change equations (4.14) and (4.15) into the general covariance form by replacing all derivatives by covariant ones:

$$F^{\mu\nu}_{;\mu} = -J^{\nu} \tag{4.16}$$

$$F_{\mu\nu;\lambda} + F_{\lambda\mu;\nu} + F_{\nu\lambda;\mu} = 0 \tag{4.17}$$

Now indices should rise and fall by means of $G_{\mu\nu}$, but not $\eta_{\alpha\beta}$, i.e.,

$$F_{\lambda\varkappa} \equiv G_{\lambda\mu} G_{\varkappa\nu} F^{\mu\nu} \tag{4.18}$$

where $G_{\lambda\mu}$ is the stochastic metric given by expressions (2.12) and (2.13). As in the usual theory, in our case electromagnetic stresses $F^{\mu\nu}$ and $F_{\mu\nu}$ in gravitational fields are antisymmetric and therefore, by using formulas (3.45) and (3.46), we can write the *Maxwell equations* in the form

$$\partial \sqrt{G}\, F^{\mu\nu}/\partial z^{\mu} = -\sqrt{G}\, J^{\nu} \tag{4.19}$$

$$\partial F_{\mu\nu}/\partial z^{\lambda} + \partial F_{\lambda\mu}/\partial z^{\nu} + \partial F_{\nu\lambda}/\partial z^{\mu} = 0 \tag{4.20}$$

Equations (4.16) and (4.17) are valid in the absence of gravity and are generally covariant. Therefore, according to the general covariance principle, they are valid in arbitrary gravitational fields.

Now we define an *electromagnetic force* acting on a particle with the charge e. In the absence of gravity it has the usual form

$$f_f^\alpha = e\tilde{F}_\beta^\alpha(d\xi^\beta/d\tau) \tag{4.21}$$

from which it follows immediately that in an arbitrarily chosen system of coordinates the electromagnetic force in any gravitational field is

$$f^\mu = eF_\nu^\mu(dz^\nu/d\tau) \tag{4.22}$$

where

$$F_\nu^\mu \equiv G_{\nu\lambda}F^{\mu\lambda} \tag{4.23}$$

It is easy to see that formula (4.22) is written in the right form, since, according to the general covariance principle, equation (4.22) is reduced to (4.21) in the local-inertial system of Minkowski coordinates and is the general covariant. Moreover, f^μ and $dz^\nu/d\tau$ are vectors and F_ν^μ is defined as a tensor.

As a calculation example, we write *Maxwell's equations* in a given constant fictitious "gravitational" field $\varepsilon_{\mu\nu}(x)$ in the three-dimensional form. For this, let us introduce the three-dimensional vectors **E** and **B** connected with components of the covariant tensor $F_{\mu\nu}$ in the same form as in Minkowski coordinates ξ^α:

$$F_{12} = B_z, \qquad F_{13} = -B_y, \qquad F_{23} = B_x$$

$$F_{10} = E_x, \qquad F_{20} = E_y, \qquad F_{30} = E_z$$

where we have used the simple connections $F^{00} = F_{00}$ and $F^{0i} = -F_{0i}$ ($i = 1, 2, 3$) between components of co- and contravariant tensors. Analogously, we link components $(-g_{00}^{(s)})^{1/2}F^{ik}$ with the components of vectors which we denote by **D** and **H**. By simple algebraic transformations one can then represent the connection $F^{\mu\nu} = g_{(s)}^{\mu\lambda}g_{(s)}^{\nu\rho}F_{\lambda\rho}$ in the form of two vector relations:

$$\mathbf{D} = h^{-1/2}\mathbf{E} + [\mathbf{H}\times\mathbf{g}], \qquad \mathbf{B} = h^{-1/2}\mathbf{H} + [\mathbf{g}\times\mathbf{E}] \tag{4.24}$$

With these definitions the four-equations (4.19) and (4.20) can be written in the form of three-dimensional equations

$$\operatorname{rot}\mathbf{E} = -(1/c)\,\partial\mathbf{B}/\partial t, \qquad \operatorname{div}\mathbf{D} = 0$$

$$\operatorname{rot}\mathbf{H} = (1/c)\,\partial\mathbf{D}/\partial t, \qquad \operatorname{div}\mathbf{B} = 0$$

in which vector operations are performed in the three-dimensional space with metric γ_{ij} given by (1.34). In expression (4.24) we have used the following notations:

$$h = -g_{00}^{(s)} \qquad \text{and} \qquad \mathbf{g} = g^i = -g_{0i}^{(s)}/g_{00}^{(s)}$$

An averaging procedure for these quantities was given in Section 1.4, from which it is easily seen that $\langle \mathbf{g} \rangle = 0$. Thus, in the whole space-time obtained by using the averaging procedure over one with stochastic metric $g^{(s)}_{\mu\nu}$, there does not exist an isolated direction connected with the "arrow" of the vector **g**, i.e., all its arrows have equal rights.

Now we calculate the *current vector* J^ν in space-time with the stochastic metric $G_{\mu\nu}$. In the special theory of relativity it has the standard form

$$j^\alpha = \sum_n e_n \int \delta^{(4)}(\xi - \xi_n)\, d\xi^\alpha_n \tag{4.25}$$

where integration is carried out along the trajectory of the nth particle. In an arbitrary system of coordinates, the four-dimensional *δ-function* is introduced in the following manner:

$$\int d^4z\, \phi(z)\, \delta^{(4)}(z - z_1) = \phi(z_1)$$

Since $G^{1/2} d^4 z$ is scalar, then the combination $G^{-1/2}\delta^{(4)}(z - z_1)$ should also be scalar, which is reduced to the usual δ-function within the special theory of relativity, where $G = 1$. Thus, the covariant vector which becomes j^α in the absence of gravity is

$$J^\mu(z) = G^{-1/2}(z) \sum_n e_n \int \delta^{(4)}(z - z_n)\, dz^\mu_n \tag{4.26}$$

Let us calculate its average value in the fictitious "gravitational" field $\varepsilon_{\mu\nu}(x)$. In this case, in (4.26) it should be replaced by $G_{\mu\nu} \to g^{(s)}_{\mu\nu}(x)$, the latter is given by (1.11). Thus, (4.26) takes the form

$$J^\mu(x) = g^{-1/2}(x) \sum_n e_n \int \delta^{(4)}(x - x_n)\, dx^\mu_n \tag{4.27}$$

To average this expression, consider the following chain identities:

$$\begin{aligned} I &= g^{-1/2}(x)\, dx^\mu_n = g^{-1/2}(x)\, d\tau (dx^\mu_n / d\tau) = g^{-1/2}\, d\tau\, U^\mu_n \\ &= g^{-1/2}\, d\tau (\partial x^\mu / \partial \xi^\rho) u^\rho_n \end{aligned}$$

where u^ρ_n is the velocity of the nth particle in the local-inertial system of reference. Further, making use of definitions (1.12), (1.28), and (3.9) in the *weak-field limit*, we obtain

$$\begin{aligned} \langle I \rangle_s = dt \langle \{ & [\delta^\mu_\rho + \delta^\mu_\rho E^2(x) - \tfrac{1}{2}\varepsilon^\mu_\rho(x_n) + \tfrac{3}{4}\varepsilon^\varkappa_\rho(x_n)\varepsilon^\mu_\varkappa(x_n)] \\ & \times [A^{1/2} - \tfrac{1}{2}A^{-1/2}(\varepsilon_{\varkappa\delta}(x_n) + \tfrac{1}{4}\varepsilon^\rho_\varkappa(x_n)\varepsilon_{\delta\rho}(x_n)) u^\varkappa_n u^\delta_n \\ & - \tfrac{1}{8}A^{-3/2}\varepsilon_{\varkappa\delta}(x_n)\varepsilon_{\rho\nu}(x_n) u^\varkappa_n u^\delta_n u^\rho_n u^\nu_n]\} \rangle \end{aligned} \tag{4.28}$$

where

$$E^2(x) = \tfrac{1}{8}\,\mathrm{Tr}\,\varepsilon'^2 + \tfrac{1}{4}\,\mathrm{Tr}\,\varepsilon^2 = \tfrac{1}{8}\varepsilon^\rho_\mu(x)\varepsilon_{\mu\rho}(x) + \tfrac{1}{4}\varepsilon_{\mu\nu}(x)\varepsilon_{\nu\mu}(x)$$

The averaging procedure in (4.28) is easily carried out by using the definition of taking the trace of the matrix $\varepsilon_{\mu\nu}(x)$ and its covariance (correlation function). For example, in the pseudo-Euclidean case (1.30) and the Euclidean case (1.77), the quantities

$$\langle \mathrm{Tr}\, \varepsilon'^2 \rangle_\varepsilon = D^\rho_{\mu,\rho\mu}(0) \qquad \text{and} \qquad \langle \mathrm{Tr}\, \varepsilon^2 \rangle_\varepsilon = D_{\mu\nu,\nu\mu}(0)$$

acquire the forms

$$\langle \mathrm{Tr}\, \varepsilon'^2 \rangle_\varepsilon = 5\tilde{D}(0), \qquad \langle \mathrm{Tr}\, \varepsilon^2 \rangle_\varepsilon = \tfrac{10}{3}\tilde{D}(0)$$

and

$$\langle \mathrm{Tr}\, \varepsilon'^2 \rangle_\varepsilon = 10\tilde{D}(0), \qquad \langle \mathrm{Tr}\, \varepsilon^2 \rangle_\varepsilon = 10\tilde{D}(0)$$

respectively. Here the function $\tilde{D}(0)$ is defined by (1.31). The result reads

$$\langle I \rangle_s = \begin{cases} dx^\mu_n[1+\frac{25}{12}\tilde{D}(0)] & \text{for the pseudo-Euclidean covariance} \\ dx^\mu_n[1+\frac{35}{8}\tilde{D}(0)] & \text{for the Euclidean covariance} \end{cases}$$

and therefore, the corresponding *averaged electromagnetic current* (4.27) is given by the simple formula

$$J^\mu(x) = j^\mu(x) \times \begin{cases} [1+\frac{25}{12}\tilde{D}(0)] \\ [1+\frac{35}{8}\tilde{D}(0)] \end{cases} \tag{4.29}$$

where $j^\alpha(x)$ is the electromagnetic current in the special theory of relativity. Expression (4.29) may be understood as a change of the charge value of the nth particle:

$$e_n \Rightarrow e'_n = e_n[1+\tfrac{35}{8}\tilde{D}(0)] \qquad \text{or} \qquad e'_n = e_n[1+\tfrac{25}{12}\tilde{D}(0)] \tag{4.30}$$

depending on taking a concrete form for the Euclidean (or pseudo-Euclidean) covariance of the stochastic field $\varepsilon_{\mu\nu}(x)$.

Thus, we see that in the fictitious "gravitational" background field $\varepsilon_{\mu\nu}(x)$, along with the value of the particle mass (see Section 1.4) its electric charge also undergoes a slight change.

Finally, notice that the *conservation law* $\partial j^\alpha/\partial \xi^\alpha = 0$ of the special theory of relativity in the scheme with a stochastic metric has the form $J^\mu_{;\mu} = 0$, or, in accordance with (3.42),

$$\partial(G^{1/2}J^\mu)/\partial z^\mu = 0 \tag{4.31}$$

The multiplier $G^{-1/2}$ in (4.26) is to compensate for the $G^{1/2}$ in (4.31), so that (4.31) expresses the constancy of the electric charge e_n.

4.3. The Energy-Momentum Tensor

In space-time with a stochastic metric the construction of the *energy-momentum tensor* is not difficult. It is achieved by using the standard method of the general theory of relativity. As usual, density and energy-momentum flow are combined in the symmetric tensor $t^{\alpha\beta}$ satisfying the conservation law

$$\partial t^{\alpha\beta}/\partial\xi^{\alpha} = q^{\beta} \tag{4.32}$$

where q^{β} is the density of the external force f^{β} acting on our system. In an isolated system $q^{\beta} = 0$. Let us define $T^{\mu\nu}$ and Q^{β} as contravariant tensors which coincide with the corresponding quantities $t^{\alpha\beta}$ and q^{β} in the absence of gravity with a stochastic metric $G_{\mu\nu}$. Then the general covariant equation coordinated with (4.32) in the case of the local-inertial system of reference has the form

$$T^{\mu\nu}_{;\mu} = Q^{\nu} \tag{4.33}$$

or, in, accordance with (3.44),

$$G^{-1/2}\,\partial(G^{1/2}T^{\mu\nu})/\partial z^{\mu} = Q^{\nu} - \Gamma^{\nu}_{\mu\lambda}T^{\mu\lambda} \tag{4.34}$$

The second term in the right-hand side of (4.34) represents the *density of the gravitational force*. As would be expected, this force acts on a system and at the same time depends only on the given system through its energy-momentum tensor. It is well known that the coefficient $(G)^{1/2}$ in (4.34) results from the fact that $(G)^{1/2}d^4z$ is the invariant volume in space-time with the stochastic metric $G_{\mu\nu}$.

The *energy-momentum tensor* of particles in the special theory of relativity is given by

$$t^{\alpha\beta} = \sum_n m_n \int (d\xi_n^{\alpha}/d\tau)\, d\xi_n^{\beta}\, \delta^{(4)}(\xi - \xi_n) \tag{4.35}$$

where integration is carried out along the particle's trajectory. By analogy with the definition of the electromagnetic current J^{μ}, we conclude that a contravariant tensor coordinated with (4.35) in the case where gravity is absent is naturally defined as

$$T^{\mu\nu} = G^{-1/2}\sum_n m_n \int (dz_n^{\mu}/d\tau)\, dz_n^{\nu}\, \delta^{(4)}(z - z_n) \tag{4.36}$$

In the case of a background radiation stochastic field $\varepsilon_{\mu\nu}(x)$, expression (4.36) takes the form

$$T^{\alpha\beta} = g_{(s)}^{-1/2}\sum_n m_n \int (dx_n^{\alpha}/d\tau)\, dx_n^{\beta}\, \delta^{(4)}(x - x_n) \tag{4.37}$$

for which the averaging procedure can easily be carried out.

Now we calculate the *energy-momentum tensor of the electromagnetic field* $F^{\alpha\beta}$. Its form in the special theory of relativity is

$$t^{\alpha\beta} = F^{\alpha}_{\gamma}F^{\beta\gamma} - \tfrac{1}{4}\eta^{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta} \tag{4.38}$$

It is not difficult to verify that the contravariant tensor coinciding with (4.38) in the absence of gravity is

$$T^{\mu\nu} = F^{\mu}_{\lambda} F^{\nu\lambda} - \tfrac{1}{4} G^{\mu\nu} F_{\lambda\varkappa} F^{\lambda\varkappa} \tag{4.39}$$

For a system consisting of particles and radiation, the energy-momentum tensor is formed of two parts, (4.36) and (4.39). Returning to *energy-momentum tensor* (4.36), *for matter* only, one easily calculates its integral form

$$\int T^{\mu 0} G^{1/2} \, d^3 z = \sum_n m_n (dz^{\mu}_n / d\tau)$$

where the sum involves all particles in the volume over which integration is carried out. It assumes that one needs to regard $T^{\mu 0} \cdot G^{1/2}$ as the spatial density of energy-momentum. From this, in particular, one can find the energy, momentum, and angular momentum for an arbitrary system:

$$P^{\mu} \equiv \int T^{\mu 0} G^{1/2} \, d^3 z \tag{4.40}$$

$$J^{\mu\nu} \equiv \int (z^{\mu} T^{\nu 0} - z^{\nu} T^{\mu 0}) G^{1/2} \, d^3 z \tag{4.41}$$

However, these quantities are not covariant tensors and are not conserved, since $T^{\mu\nu} G^{1/2}$ is not preserved, i.e., $\partial(T^{\mu\nu} G^{1/2})/\partial z^{\mu}$ does not become zero, due to the fact that the exchange of energy and momentum between matter and gravity takes place.

4.4. Hydrodynamics and Hydrostatics

In the absence of gravity, the energy-momentum tensor of an ideal liquid is given by the following formula [for details, see Weinberg (1972)]:

$$t^{\alpha\beta} = p \eta^{\alpha\beta} + (p + \rho) u^{\alpha} u^{\beta} \tag{4.42}$$

where u^{α} is the four-velocity of the liquid and $u^0 = (1 - \mathrm{v}^2/c^2)^{-1/2}$, $\mathbf{u} = \mathbf{v} u^0$. The contravariant tensor, which is reduced to (4.42) in the absence of gravity, reads

$$T^{\mu\nu} = p G^{\mu\nu} + (p + \rho) U^{\mu} U^{\nu} \tag{4.43}$$

where U^{μ} is the local value $dz^{\mu}/d\tau$ for the liquid element in an accompanying system of reference. Notice that p and ρ are always defined as the density of pressure and energy measured by an observer in the local-inertial system of reference moving together with the liquid at the moment of measurement, and are therefore scalars. Let us consider the conditions of the conservation of the energy-momentum tensor, leading to the *hydrodynamic equations*:

$$T^{\mu\nu}_{;\nu} = (\partial p/\partial z^\nu)G^{\mu\nu} + G^{-1/2}\{\partial[G^{1/2}(p+\rho)U^\mu U^\nu]/\partial z^\nu\}$$
$$+\Gamma^\mu_{\nu\lambda}(p+\rho)U^\nu U^\lambda = 0 \tag{4.44}$$

The last term in (4.44) represents the *gravitational force* acting on the system. Notice that since $\eta_{\alpha\beta}u^\alpha u^\beta = -1$ in the absence of gravity, we should write

$$G_{\mu\nu}U^\mu U^\nu = -1 \tag{4.45}$$

in the presence of gravity with the stochastic metric $G_{\mu\nu}$.

As an example, consider the case when the liquid is placed in a state of *hydrostatic equilibrium*. Since the liquid does not move, expression (4.45) leads to

$$U^0 = (-G_{00})^{-1/2}, \qquad U^\lambda = 0 \quad \text{for} \quad \lambda \neq 0$$

Moreover, all derivatives of $G_{\mu\nu}$, p, and ρ with respect to time variables disappear. In particular, we have

$$\Gamma^\mu_{00} = -\tfrac{1}{2}G^{\mu\nu}\,\partial G_{00}/\partial z^\nu$$

and

$$\partial[(p+\rho)U^\mu U^\nu]/\partial z^\nu = 0$$

Multiplying (4.44) by $G_{\mu\lambda}$, we get

$$-\partial p/\partial z^\lambda = (p+\rho)\,\partial[\ln(-G_{00})^{1/2}]/\partial z^\lambda \tag{4.46}$$

As in the usual case, this condition is trivial for $\lambda = 0$, while for spatial-like value of λ, expression (4.46) is regarded as the usual nonrelativistic condition of *hydrostatic equilibrium*, where we should put $p+\rho$ and $(-G_{00})^{1/2}$ instead of mass density and gravitational potential, respectively. Equation (4.46) is easily solved if pressure p is given as a function of ρ. The solution has the form

$$\int dp(\rho)\,[p(\rho)+\rho]^{-1} = -\ln(-G_{00})^{1/2} + \text{const} \tag{4.47}$$

For example, if the dependence of $p(\rho)$ is the *power law* $p(\rho) \sim \rho^N$, then equality (4.47) for $N \neq 1$ reads

$$(p+\rho)\rho^{-1} \sim (-G_{00})^{(1-N)/2N} \tag{4.48}$$

but for $N = 1$,

$$\rho \sim (-G_{00})^{-(p+\rho)/2p} \tag{4.49}$$

In the usual theory of gravity the latter shows that when $p=\rho/3$, gravity never supports the equilibrium of an ultrarelativistic liquid located in a finite volume, since in this case, expression (4.49) has the form

$$\rho \sim (-G_{00})^{-2}$$

Since ρ must be equal to zero outside the liquid, G_{00} is a singular function on its surface. However, in our case, when the fictitious radiation field $\varepsilon_{\mu\nu}(z)$ is strong, i.e., it is quite possible to compensate a pure gravitational external field $g^0_{\mu\nu}$ by assuming

$$g^0_{00} = -\varepsilon_{00} - \tfrac{1}{4}\varepsilon^\rho_0(x)\varepsilon_{\rho 0}(x) \approx -\tfrac{1}{4}\varepsilon^\rho_0(x)\varepsilon_{\rho 0}(x)$$

then gravity with a high fluctuation of the space-time metric may support the equilibrium of an ultrarelativistic liquid located in a finite volume, since the condition $\rho \to 0$ is achieved by means of the equality

$$g^0_{00} \sim -\tfrac{1}{4}\varepsilon^\rho_0(x)\varepsilon_{0\rho}(x) \qquad (|\varepsilon_{\mu\nu}(x)| \gg 1)$$

where the true gravitational metric $g^0_{00}(x) \neq \infty$ on its surface.

Finally, it should be noted that contributions of gravitational effects to any physical system due to a stochastic fluctuation in the space-time metric is calculated by the same method as used in the usual theory of gravity by using the general covariance principle.

5. MODIFIED EINSTEIN EQUATION IN SPACE-TIME WITH STOCHASTIC METRIC

In this section we reconstruct Einstein's equation from the point of view of a stochastic fluctuation of the space-time metric. Here our goal is to find a gravitational field equation written in the general covariant form by using the equivalence principle for gravity itself. Before obtaining the corresponding field equation one must form the curvature tensor by means of the stochastic metric and carry out some tensor algebraic operations. Now we turn to these complex problems.

5.1. Redefinition of the Curvature Tensor

In accordance with the usual theory of gravity, we first construct a tensor from the stochastic metric tensor and its first and second derivatives in space-time with a stochastic metric. In order to do this, we recall the transformation rule of the affine connection (see Section 3),

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial z^\lambda}{\partial z'^\tau}\frac{\partial z'^\rho}{\partial z^\mu}\frac{\partial z'^\sigma}{\partial z^\nu}\Gamma'^\tau_{\rho\sigma} + \frac{\partial z^\lambda}{\partial z'^\tau}\frac{\partial^2 z'^\tau}{\partial z^\mu\,\partial z^\nu} \tag{5.1}$$

This is just relation (3.10) in which the primed and unprimed coordinates are rearranged. On the right-hand side of (5.1) there is a nonhomogeneity damaging the tensor character of $\Gamma^\lambda_{\mu\nu}(z)$, and therefore, we attempt to separate it:

$$\frac{\partial^2 z'^\tau}{\partial z^\mu \partial z^\nu} = \frac{\partial z'^\tau}{\partial z^\lambda}\Gamma^\lambda_{\mu\nu} - \frac{\partial z'^\rho}{\partial z^\mu}\frac{\partial z'^\sigma}{\partial z^\nu}\Gamma'^\tau_{\rho\sigma} \tag{5.2}$$

In order to avoid the left part, we use the noncommutability of partial derivatives. Differentiation over $z^\varkappa$ gives

$$\begin{aligned}\frac{\partial^3 z'^\tau}{\partial z^\varkappa \partial z^\mu \partial z^\nu} = {} & \Gamma^\lambda_{\mu\nu}\left(\frac{\partial z'^\tau}{\partial z^\eta}\Gamma^\eta_{\varkappa\lambda} - \frac{\partial z'^\rho}{\partial z^\varkappa}\frac{\partial z'^\sigma}{\partial z^\lambda}\Gamma'^\tau_{\rho\sigma}\right)\\ & - \Gamma'^\tau_{\rho\sigma}\frac{\partial z'^\rho}{\partial z^\mu}\left(\frac{\partial z'^\sigma}{\partial z^\eta}\Gamma^\eta_{\varkappa\nu} - \frac{\partial z'^\eta}{\partial z^\varkappa}\frac{\partial z'^\xi}{\partial z^\nu}\Gamma'^\sigma_{\eta\xi}\right)\\ & - \Gamma'^\tau_{\rho\sigma}\frac{\partial z'^\sigma}{\partial z^\nu}\left(\frac{\partial z'^\rho}{\partial z^\eta}\Gamma^\eta_{\varkappa\mu} - \frac{\partial z'^\eta}{\partial z^\varkappa}\frac{\partial z'^\xi}{\partial z^\mu}\Gamma'^\rho_{\eta\xi}\right)\\ & + \frac{\partial z'^\tau}{\partial z^\lambda}\frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial z^\varkappa} - \frac{\partial z'^\rho}{\partial z^\mu}\frac{\partial z'^\sigma}{\partial z^\nu}\frac{\partial z'^\eta}{\partial z^\varkappa}\frac{\partial \Gamma'^\tau_{\rho\sigma}}{\partial z'^\eta}\end{aligned}$$

Further, collecting similar terms and rearranging some indices, we get

$$\begin{aligned}\frac{\partial^3 z'^\tau}{\partial z^\varkappa \partial z^\mu \partial z^\nu} = {} & \frac{\partial z'^\tau}{\partial z^\lambda}\left(\frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial z^\varkappa} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\varkappa\eta}\right)\\ & - \frac{\partial z'^\rho}{\partial z^\mu}\frac{\partial z'^\sigma}{\partial z^\nu}\frac{\partial z'^\eta}{\partial z^\varkappa}\left(\frac{\partial \Gamma'^\tau_{\rho\sigma}}{\partial z'^\eta} - \Gamma'^\tau_{\rho\lambda}\Gamma'^\lambda_{\eta\sigma} - \Gamma'^\tau_{\lambda\sigma}\Gamma'^\lambda_{\eta\rho}\right)\\ & - \Gamma'^\tau_{\rho\sigma}\frac{\partial z'^\sigma}{\partial z^\lambda}\left(\Gamma^\lambda_{\mu\nu}\frac{\partial z'^\rho}{\partial z^\varkappa} + \Gamma^\lambda_{\varkappa\nu}\frac{\partial z'^\rho}{\partial z^\mu} + \Gamma^\lambda_{\varkappa\mu}\frac{\partial z'^\rho}{\partial z^\nu}\right)\end{aligned} \tag{5.3}$$

Rearranging indices ν and $\varkappa$ and subtracting the obtained result from (5.3), we see that all terms involving the product of Γ and Γ' disappear and the following expression remains:

$$\begin{aligned}0 = {} & \frac{\partial z'^\tau}{\partial z^\lambda}\left(\frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial z^\varkappa} - \frac{\partial \Gamma^\lambda_{\mu\varkappa}}{\partial z^\nu} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\varkappa\eta} - \Gamma^\eta_{\mu\varkappa}\Gamma^\lambda_{\nu\eta}\right)\\ & - \frac{\partial z'^\rho}{\partial z^\mu}\frac{\partial z'^\sigma}{\partial z^\nu}\frac{\partial z'^\eta}{\partial z^\varkappa}\\ & \times\left(\frac{\partial \Gamma'^\tau_{\rho\sigma}}{\partial z'^\eta} - \frac{\partial \Gamma'^\tau_{\rho\eta}}{\partial z'^\sigma} - \Gamma'^\tau_{\lambda\sigma}\Gamma'^\lambda_{\eta\rho} + \Gamma'^\tau_{\lambda\eta}\Gamma'^\lambda_{\sigma\rho}\right)\end{aligned}$$

It can be rewritten in the form of the transformation

$$R'^{\tau}_{\rho\sigma\eta} = \frac{\partial z'^{\tau}}{\partial z^{\lambda}} \frac{\partial z^{\mu}}{\partial z'^{\rho}} \frac{\partial z^{\nu}}{\partial z'^{\sigma}} \frac{\partial z^{\varkappa}}{\partial z'^{\eta}} R^{\lambda}_{\mu\nu\varkappa} \tag{5.4}$$

where

$$R^{\lambda}_{\mu\nu\varkappa} \equiv \partial\Gamma^{\lambda}_{\mu\nu}/\partial z^{\varkappa} - \partial\Gamma^{\lambda}_{\mu\varkappa}/\partial z^{\nu} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\varkappa\eta} - \Gamma^{\eta}_{\mu\varkappa}\Gamma^{\lambda}_{\nu\eta} \tag{5.5}$$

Equation (5.4) asserts that $R^{\lambda}_{\mu\nu\varkappa}$ is a tensor; we call it the *Riemann-Christoffel curvature tensor* defined by using a stochastic metric. The tensor (5.5) constructed in this way is unique. For the proof of this, see, for example, Weinberg (1972).

In the limit of a *weak field* $\varepsilon_{\mu\nu}(z)$ or small fluctuation of the space-time metric, one can average (5.5) by using definition (2.24) for the affine connection $\Gamma^{\lambda}_{\mu\nu}(z)$, where the stochastic metric $G_{\mu\nu}(z)$ is given by formula (2.13). To carry out the averaging procedure for (5.5), we first define the explicit form of $\Gamma^{\lambda}_{\mu\nu}$, that is

$$\begin{aligned}\Gamma^{\lambda}_{\mu\nu}(z) = \Gamma^{0\lambda}_{\mu\nu} - \tfrac{1}{2}\varepsilon^{\rho\lambda}(z)\gamma_{\rho;\mu\nu} + \tfrac{3}{8}\varepsilon^{\rho\delta}(z)\varepsilon^{\lambda}_{\delta}(z)\gamma_{\rho;\mu\nu} + \tfrac{1}{2}g_0^{\rho\lambda}\mathscr{E}_{\rho;\mu\nu} \\ - \tfrac{1}{2}\varepsilon^{\rho\lambda}(z)\mathscr{E}_{\rho;\mu\nu} + \tfrac{1}{8}g_0^{\rho\lambda}E_{\rho g;\mu\nu} + O(\varepsilon^3)\end{aligned} \tag{5.6}$$

where $\Gamma^{0\lambda}_{\mu\nu}(z)$ and $g_0^{\rho\lambda}(z)$ are the usual gravitational affine connection and the metric tensor, respectively. In (5.6) we have used the following notation:

$$\begin{aligned}\gamma_{\rho;\mu\nu}(z) &= \partial g^0_{\rho\nu}/\partial z^{\mu} + \partial g^0_{\mu\rho}/\partial z^{\nu} - \partial g^0_{\mu\nu}/\partial z^{\rho} \\ \mathscr{E}_{\rho;\mu\nu}(z) &= \partial\varepsilon_{\nu\rho}/\partial z^{\mu} + \partial\varepsilon_{\mu\rho}/\partial z^{\nu} - \partial\varepsilon_{\nu\mu}/\partial z^{\rho} \\ E_{\rho;\mu\nu}(z) &= \partial(\varepsilon^{\varkappa}_{\nu}\varepsilon_{\rho\varkappa})/\partial z^{\mu} + \partial(\varepsilon^{\varkappa}_{\mu}\varepsilon_{\rho\varkappa})/\partial z^{\nu} - \partial(\varepsilon^{\varkappa}_{\mu}\varepsilon_{\nu\varkappa})/\partial z^{\rho}\end{aligned} \tag{5.7}$$

Assuming $\langle\partial\varepsilon^{\varkappa}_{\nu}(z)/\partial z^{\mu}\cdot\varepsilon_{\rho\varkappa}\rangle_{\varepsilon} = 0$ for the background radiation field $\varepsilon_{\mu\nu}(z)$ without a particle, we find

$$\langle\Gamma^{\lambda}_{\mu\nu}(z)\rangle_s = \Gamma^{0\lambda}_{\mu\nu}(z) + \tfrac{3}{8}D^{\rho\delta,\lambda}_{,\delta}(0)\gamma_{\rho;\mu\nu} \tag{5.8}$$

where

$$D^{\rho\delta,\lambda}_{,\rho}(0) = \tfrac{5}{9}(\delta^{\rho}_{\rho}\delta^{\lambda\delta} + \delta^{\rho\lambda}\delta^{\delta}_{\rho} - \tfrac{1}{2}\delta^{\rho\delta}\delta^{\lambda}_{\rho})\tilde{D}(0) = \tfrac{5}{2}\delta^{\delta\lambda}\tilde{D}(0)$$

for the Euclidean procedure of taking the covariances of field $\varepsilon_{\mu\nu}(z)$. The averaging $R^{\lambda}_{\mu\nu\varkappa}(z)$ with the combination of the metric tensor $G_{\mu\nu}(z)$ will be given in Section 5.3.

5.2. Ricci Tensor and the Scalar Curvature

By using the metric tensor and linear combinations of the curvature tensor $R^{\lambda}_{\mu\nu\varkappa}$ one can construct other tensor quantities. Among them the contracted forms are most important:

1. The *Ricci tensor*

$$R_{\mu\varkappa} \equiv R^{\lambda}_{\mu\lambda\varkappa} \tag{5.9}$$

2. The *scalar curvature*

$$R = G^{\mu\varkappa} R_{\mu\varkappa} \tag{5.10}$$

To carry out the averaging procedure for these tensors it is useful to express $R^{\lambda}_{\mu\nu\varkappa}$ through the second derivative of the stochastic metric tensor $G_{\mu\nu}$. For this purpose we consider its covariant version $R_{\lambda\mu\nu\varkappa} \equiv G_{\lambda\sigma} R^{\sigma}_{\mu\nu\varkappa}$. Taking into account definitions (5.5) and (2.24), we get

$$\begin{aligned} R_{\lambda\mu\nu\varkappa} &= \tfrac{1}{2} G_{\lambda\sigma}(\partial G^{\sigma\rho} \gamma_{\rho;\mu\nu}/\partial z^{\varkappa}) - \tfrac{1}{2} G_{\lambda\sigma}(\partial G^{\sigma\rho}\gamma_{\rho;\varkappa\mu}/\partial z^{\nu}) \\ &\quad + G_{\lambda\sigma}(\Gamma^{\eta}_{\mu\nu}\Gamma^{\sigma}_{\varkappa\eta} - \Gamma^{\eta}_{\mu\varkappa}\Gamma^{\sigma}_{\nu\eta}) \end{aligned} \tag{5.11}$$

Further, by using the identity $G^{\rho\delta}G_{\delta\lambda} = \delta^{\rho}_{\lambda}$, one can easily ensure that

$$G_{\lambda\rho}\, \partial G^{\rho\sigma}/\partial z^{\varkappa} = -G^{\sigma\rho}\, \partial G_{\lambda\sigma}/\partial z^{\varkappa} = -G^{\sigma\rho}(\Gamma^{\eta}_{\varkappa\lambda} G_{\eta\sigma} + \Gamma^{\eta}_{\varkappa\sigma} G_{\eta\lambda})$$

With this formula, expression (5.11) takes the form

$$\begin{aligned} R_{\lambda\mu\nu\varkappa} &= \tfrac{1}{2} N_{\lambda\mu\nu\varkappa} - (\Gamma^{\eta}_{\varkappa\lambda} G_{\eta\sigma} + \Gamma^{\eta}_{\varkappa\sigma} G_{\eta\lambda})\Gamma^{\sigma}_{\mu\nu} + (\Gamma^{\eta}_{\nu\lambda} G_{\eta\sigma} + \Gamma^{\eta}_{\nu\sigma} G_{\eta\lambda})\Gamma^{\sigma}_{\mu\varkappa} \\ &\quad + G_{\lambda\sigma}(\Gamma^{\eta}_{\mu\nu}\Gamma^{\sigma}_{\varkappa\eta} - \Gamma^{\eta}_{\mu\varkappa}\Gamma^{\sigma}_{\nu\eta}) \end{aligned}$$

where

$$\begin{aligned} N_{\lambda\mu\nu\varkappa} &= \partial^2 G_{\lambda\nu}/\partial z^{\varkappa}\, \partial z^{\mu} - \partial^2 G_{\mu\nu}/\partial z^{\varkappa}\, \partial z^{\lambda} \\ &\quad - \partial^2 G_{\lambda\varkappa}/\partial z^{\nu}\, \partial z^{\mu} + \partial^2 G_{\mu\varkappa}/\partial z^{\nu}\, \partial z^{\lambda} \end{aligned} \tag{5.12}$$

Most terms of the $\Gamma\Gamma$ type mutually cancel, the result reads

$$R_{\lambda\mu\nu\varkappa} = \tfrac{1}{2} N_{\lambda\mu\nu\varkappa} + G_{\eta\sigma}(\Gamma^{\eta}_{\nu\lambda}\Gamma^{\sigma}_{\mu\varkappa} - \Gamma^{\eta}_{\varkappa\lambda}\Gamma^{\sigma}_{\mu\nu}) \tag{5.13}$$

As in the usual theory of gravity, from (5.13) it is easy to see the algebraic properties of the curvature tensor:

1. Symmetry

$$R_{\lambda\mu\nu\varkappa} = R_{\nu\varkappa\lambda\mu} \tag{5.14}$$

2. Antisymmetry

$$R_{\lambda\mu\nu\varkappa} = -R_{\mu\lambda\nu\varkappa} = -R_{\lambda\mu\varkappa\nu} = R_{\mu\lambda\varkappa\nu} \tag{5.15}$$

3. Cyclicality

$$R_{\lambda\mu\nu\varkappa} + R_{\lambda\varkappa\mu\nu} + R_{\lambda\nu\varkappa\mu} = 0 \tag{5.16}$$

The property of symmetry (5.14) shows that the Ricci tensor

$$R_{\mu\varkappa} = G^{\lambda\nu} R_{\lambda\mu\nu\varkappa} \tag{5.17}$$

is symmetric

$$R_{\mu\varkappa} = R_{\varkappa\mu}$$

and the antisymmetry property (5.15) asserts that $R_{\mu\varkappa}$ is unique. Indeed, multiplying (5.15) by the quantities $G^{\lambda\nu}$, $G^{\lambda\mu}$, and $G^{\nu\varkappa}$, we get

$$R_{\mu\varkappa} = -G^{\lambda\nu}R_{\mu\lambda\nu\varkappa} = -G^{\lambda\nu}R_{\lambda\mu\varkappa\nu} = G^{\lambda\nu}R_{\mu\lambda\varkappa\nu}$$

$$G^{\lambda\mu}R_{\lambda\mu\nu\varkappa} = G^{\nu\varkappa}R_{\lambda\mu\nu\varkappa} = 0$$

From the antisymmetry condition (5.15) we see that there exists only one possibility contracting $R_{\lambda\mu\nu\varkappa}$ for obtaining scalar:

$$R \equiv G^{\lambda\nu}G^{\mu\varkappa}R_{\lambda\mu\nu\varkappa} = -G^{\lambda\nu}G^{\mu\varkappa}R_{\mu\lambda\nu\varkappa}$$

$$0 = G^{\lambda\mu}G^{\nu\varkappa}R_{\lambda\mu\nu\varkappa}$$

The condition (5.16) excludes another scalar which would be formed in the four-dimensional space:

$$G^{-1/2}\varepsilon^{\lambda\mu\nu\varkappa}R_{\lambda\mu\nu\varkappa} = 0$$

5.3. Averaging Procedure in Space-Time with the Stochastic Metric $G_{\mu\nu}$

Now we are able to average the Ricci tensor $R_{\mu\varkappa}$ and the combination $G_{\mu\nu}R$ in space-time with a small stochastic metric $G_{\mu\nu}$. First, we do this for $R_{\mu\varkappa}(z)$. Inserting definitions (2.13), (2.22), and (5.13) into equality (5.17) and carrying out some elementary calculations, we have

$$R_{\mu\varkappa} = R^0_{\mu\varkappa} + Q_{\mu\varkappa} + M_{\mu\varkappa} \tag{5.18}$$

where $R^0_{\mu\varkappa}$ is the usual Ricci tensor defined by the metric tensor $g^0_{\mu\nu}$ for an external gravitational field. The additional quantities $M_{\mu\varkappa}$ and $Q_{\mu\varkappa}$ are proportional to the field $\varepsilon_{\mu\nu}(z)$ and its squared values:

$$\begin{aligned} M_{\mu\varkappa} &= -\tfrac{1}{2}\varepsilon^{\lambda\nu}N^1_{\mu\lambda\nu\varkappa} + \tfrac{1}{2}g^{0\lambda\nu}N^2_{\mu\lambda\nu\varkappa} + g^{0\nu\lambda}g^0_{\eta\sigma}\Lambda^{\eta\sigma}_{1\mu\lambda\nu\varkappa} \\ &\quad + (-\varepsilon^{\lambda\nu}g^0_{\eta\sigma} + \varepsilon_{\eta\sigma}g^{0\lambda\nu})(\Gamma^{0\eta}_{\nu\lambda}\Gamma^{0\sigma}_{\mu\varkappa} - \Gamma^{0\eta}_{\varkappa\lambda}\Gamma^{0\sigma}_{\mu\nu}) \end{aligned} \tag{5.19a}$$

$$\begin{aligned} Q_{\mu\varkappa} &= \tfrac{3}{4}\varepsilon^{\lambda\delta}\varepsilon^{\nu}_{\delta}N^1_{\mu\lambda\nu\varkappa} - \tfrac{1}{2}\varepsilon^{\lambda\nu}N^2_{\mu\lambda\nu\varkappa} + \tfrac{1}{8}g^{0\lambda\nu}N^3_{\mu\lambda\nu\varkappa} \\ &\quad + (-\varepsilon^{\lambda\nu}g^0_{\eta\sigma} + \varepsilon_{\eta\sigma}g^{0\lambda\nu})\Lambda^{\eta\sigma}_{1\mu\lambda\nu\varkappa} \\ &\quad + (\tfrac{3}{4}\varepsilon^{\lambda\delta}\varepsilon^{\nu}_{\delta}g^0_{\eta\sigma} - \varepsilon^{\lambda\nu}\varepsilon_{\eta\sigma} + \tfrac{1}{4}g^{0\lambda\nu}\varepsilon^{\gamma}_{\eta}\varepsilon_{\sigma\gamma}) \\ &\quad \times (\Gamma^{0\eta}_{\nu\lambda}\Gamma^{0\sigma}_{\mu\varkappa} - \Gamma^{0\eta}_{\varkappa\lambda}\Gamma^{0\sigma}_{\mu\nu}) + g^{0\lambda\nu}g^0_{\eta\sigma}\Lambda^{\eta\sigma}_{2\mu\lambda\nu\varkappa} \end{aligned} \tag{5.19b}$$

where

$$N^i_{\mu\lambda\nu\varkappa} = \frac{\partial^2 I^i_{\lambda\nu}}{\partial z^\varkappa\,\partial z^\mu} - \frac{\partial^2 I^i_{\mu\nu}}{\partial z^\varkappa\,\partial z^\lambda} - \frac{\partial^2 I^i_{\lambda\varkappa}}{\partial z^\nu\,\partial z^\mu} + \frac{\partial^2 I^i_{\mu\varkappa}}{\partial z^\nu\,\partial z^\lambda}, \qquad i = 1, 2, 3 \tag{5.20}$$

$$I^1_{\mu\nu} = g^0_{\mu\nu}, \qquad I^2_{\mu\nu} = \varepsilon_{\mu\nu}, \qquad I^3_{\mu\nu} = \varepsilon^{\rho}_{\mu}\varepsilon_{\nu\rho}$$

$$\begin{aligned}\Lambda^{\eta\sigma}_{1\mu\lambda\nu\varkappa} &= -\tfrac{1}{2}\varepsilon^{\rho\eta}\Gamma^{0\sigma}_{\mu\varkappa}\gamma_{\rho;\nu\lambda} + \tfrac{1}{2}\varepsilon^{\rho\eta}\Gamma^{0\sigma}_{\mu\nu}\gamma_{\rho;\varkappa\lambda} \\ &\quad -\tfrac{1}{2}\varepsilon^{\rho\sigma}\Gamma^{0\eta}_{\nu\lambda}\gamma_{\rho;\mu\varkappa} + \tfrac{1}{2}\varepsilon^{\rho\sigma}\Gamma^{0\eta}_{\varkappa\lambda}\gamma_{\rho;\mu\nu};\end{aligned} \tag{5.21}$$

$$\begin{aligned}\Lambda^{\eta\sigma}_{2\mu\lambda\nu\varkappa} &= \tfrac{3}{8}\varepsilon^{\rho\gamma}\varepsilon^{\eta}_{\gamma}\Gamma^{0\sigma}_{\mu\varkappa}\gamma_{\rho;\nu\lambda} - \tfrac{3}{8}\varepsilon^{\rho\gamma}\varepsilon^{\eta}_{\gamma}\Gamma^{0\sigma}_{\mu\nu}\gamma_{\rho;\varkappa\lambda} \\ &\quad + \tfrac{1}{4}\varepsilon^{\rho\eta}\varepsilon^{\rho'\sigma}\gamma_{\rho;\nu\lambda}\gamma_{\rho';\mu\varkappa} - \tfrac{1}{4}\varepsilon^{\rho\eta}\varepsilon^{\rho'\sigma}\gamma_{\rho;\varkappa\lambda}\gamma_{\rho';\mu\nu} \\ &\quad + \tfrac{3}{8}\varepsilon^{\rho'\varkappa}\varepsilon^{\sigma}_{\varkappa}\Gamma^{0\eta}_{\nu\lambda}\gamma_{\rho';\mu\varkappa} - \tfrac{3}{8}\varepsilon^{\rho'\gamma}\varepsilon^{\sigma}_{\gamma}\Gamma^{0\eta}_{\varkappa\lambda}\gamma_{\rho';\mu\nu} \\ &\quad + \tfrac{1}{4}g^{0\rho\eta}g^{0\rho'\sigma}\mathscr{E}_{\rho;\nu\lambda}\mathscr{E}_{\rho';\mu\varkappa} - \tfrac{1}{4}g^{0\rho\eta}g^{0\rho'\sigma}\mathscr{E}_{\rho;\varkappa\lambda}\mathscr{E}_{\rho';\mu\nu}\end{aligned} \tag{5.22}$$

where the values of $\gamma_{\rho;\mu\nu}$ and $\mathscr{E}_{\rho;\mu\nu}$ are determined by formula (5.7).

In order to find the *covariance* of the field $\varepsilon_{\mu\nu}(x)$, we use the same method as in Section 1.2. Here we employ the *divisors*

$$d^{(1)}_{\mu\nu}(q^E) = q^E_\mu q^E_\nu - q^2_E\delta_{\mu\mu}, \qquad d^{(2)}_{\mu\nu}(q^E) = q^E_\mu q^E_\nu / q^2_E - \delta_{\mu\nu}$$

and the conditions (1.18a), (1.18b). Thus, recalling

$$\varepsilon_{\mu\nu}(z) = i^{-1}(2\pi)^{-4}\int d^4q\, e^{iqz}\tilde{\varepsilon}_{\mu\nu}(q)$$

$$\partial^2\varepsilon_{\lambda\nu}(z)/\partial z^\varkappa\,\partial z^\mu = i(2\pi)^{-4}\int d^4q\, q_\varkappa q_\mu\, e^{iqz}\tilde{\varepsilon}_{\lambda\nu}(q)$$

and

$$\langle\tilde{\varepsilon}_{\lambda\nu}(q_1)\varepsilon_{\rho\sigma}(q_2)\rangle = i(2\pi)^4\delta^{(4)}(q_1+q_2)D_{\lambda\nu,\rho\sigma}(q_1)$$

we obtain the following covariances:

$$\begin{aligned}D_{\mu\nu,\rho\sigma} &= \langle\varepsilon_{\mu\nu}(z)\varepsilon_{\rho\sigma}(z)\rangle_\varepsilon = i^{-1}(2\pi)^{-4}\int d^4q\, D_{\mu\nu,\rho\sigma}(q) \\ D^1_{\rho\delta,\lambda\nu;\varkappa\mu} &= \langle\varepsilon_{\rho\delta}(z)\cdot\partial^2\varepsilon_{\lambda\nu}(z)/\partial z^\varkappa\,\partial z^\mu\rangle_\varepsilon \\ &= i(2\pi)^{-4}\int d^4q\, q_\varkappa q_\mu D_{\rho\delta,\lambda\nu}(q)\end{aligned} \tag{5.23}$$

or in the Euclidean metric

$$\begin{aligned}D_{\mu\nu,\rho\delta} &= (2\pi)^{-4}\int d^4q_E\, D^E_{\mu\nu,\rho\delta}(q_E) \\ D^1_{\rho\delta,\lambda\nu;\varkappa\mu} &= -(2\pi)^{-4}\int d^4q_E\, q^E_\varkappa q^E_\mu D^E_{\rho\delta,\lambda\nu}(q_E)\end{aligned} \tag{5.24}$$

Here we distinguish two versions of the definition of the covariance resulting from (5.23) and (5.24), respectively. It is easy to verify that the case (5.24) leads to

$$
\begin{aligned}
D_{\mu\nu,\rho\delta}(0) &= \tfrac{5}{9}(\delta_{\mu\rho}\delta_{\nu\delta}+\delta_{\mu\delta}\delta_{\nu\rho}-\tfrac{1}{2}\delta_{\mu\nu}\delta_{\rho\delta})\tilde{D}(0)\\
D^1_{\rho\sigma,\mu\varkappa;\nu\lambda}(0) &= \{(4/3a+\tfrac{1}{6})(\delta_{\nu\lambda}\delta_{\rho\mu}\delta_{\sigma\varkappa}+\delta_{\nu\lambda}\delta_{\rho\varkappa}\delta_{\mu\sigma})\\
&\quad +(4/3a-\tfrac{1}{9})\delta_{\nu\lambda}\delta_{\rho\sigma}\delta_{\mu\varkappa}+(4/3a-\tfrac{1}{24})(\delta_{\nu\rho}\delta_{\lambda\mu}\delta_{\sigma\varkappa}+\delta_{\nu\rho}\delta_{\lambda\varkappa}\delta_{\mu\sigma}\\
&\quad +\delta_{\nu\mu}\delta_{\lambda\rho}\delta_{\sigma\varkappa}+\delta_{\nu\mu}\delta_{\lambda\sigma}\delta_{\rho\varkappa}+\delta_{\nu\sigma}\delta_{\lambda\mu}\delta_{\rho\varkappa}\\
&\quad +\delta_{\nu\sigma}\delta_{\lambda\varkappa}\delta_{\rho\mu}+\delta_{\nu\varkappa}\delta_{\lambda\rho}\delta_{\mu\sigma}+\delta_{\nu\varkappa}\delta_{\lambda\sigma}\delta_{\rho\mu})\\
&\quad +(4/3a+\tfrac{1}{36})(\delta_{\nu\rho}\delta_{\lambda\sigma}\delta_{\mu\varkappa}+\delta_{\nu\mu}\delta_{\lambda\varkappa}\delta_{\rho\sigma}\\
&\quad +\delta_{\nu\sigma}\delta_{\lambda\rho}\delta_{\mu\varkappa}+\delta_{\nu\varkappa}\delta_{\lambda\mu}\delta_{\rho\sigma})\}\tilde{D}_1(0), \qquad a=\tfrac{1}{192}
\end{aligned}
\tag{5.25}
$$

where the functions $\tilde{D}(0)$ and $\tilde{D}_1(0)$ depend on the conditions (1.18a) and (1.18b) and are defined by the formulas

$$
\tilde{D}(0)=\begin{cases} G^2(2\pi)^{-4}\int d^4q\, q^4\tilde{D}_l^{(1)}(q^2) & \text{for (1.18a)}\\ G/l^2 & \text{for (1.18b)}\end{cases}
\tag{5.26}
$$

and

$$
\tilde{D}_1(0)=\begin{cases} G^2(2\pi)^{-4}\int d^4q\, q^6\tilde{D}_l^{(1)}(q^2) & \text{for (1.18a)}\\ G(2\pi)^{-4}\int d^4q\, q^2\tilde{D}_l^{(2)}(q^2) & \text{for (1.18b)}\end{cases}
\tag{5.27}
$$

For the case (5.23) one obtains the same formula (5.25) in which the Kronecker symbol δ^{ν}_{μ} should be replaced by the Minkowski metric $\eta_{\mu\nu}$. For precision, we further use expression (5.25).

Finally, taking into account relations (5.19a), (5.19b), (5.24), and (5.25) and after some elementary but tedious calculations, we get from (5.18):

$$
\begin{aligned}
\langle R_{\mu\varkappa}\rangle_s &= R^0_{\mu\varkappa}+\tfrac{15}{8}\tilde{D}(0)N^1_{\lambda\mu\lambda\varkappa}+\tfrac{1}{2}(64/a+\tfrac{13}{6})\tilde{D}_1(0)\delta_{\mu\varkappa}+N_{\mu\varkappa}\\
&\quad +\tfrac{15}{16}A_{\mu\varkappa}-Q_{\mu\varkappa}+L_{\mu\varkappa}+M_{\mu\varkappa}, \qquad a=\tfrac{1}{192}
\end{aligned}
\tag{5.28}
$$

where

$$
\begin{aligned}
N_{\mu\varkappa} &= \tfrac{5}{18}\tilde{D}(0)(2\gamma_{\lambda;\nu\lambda}\gamma_{\nu;\mu\varkappa}-\gamma_{\rho;\varkappa\rho}\gamma_{\nu;\mu\nu}-\gamma_{\rho;\mu\nu}\gamma_{\nu;\varkappa\rho}\\
&\quad -\tfrac{1}{4}\gamma_{\rho;\lambda\lambda}\gamma_{\rho;\mu\varkappa}+\tfrac{1}{4}\gamma_{\rho;\varkappa\lambda}\gamma_{\rho;\mu\lambda})\\
A_{\mu\varkappa} &= \tilde{D}(0)(\gamma_{\sigma;\lambda\lambda}\Gamma^{0\sigma}_{\mu\varkappa}-\gamma_{\sigma;\varkappa\lambda}\Gamma^{0\sigma}_{\mu\lambda})\\
Q_{\mu\varkappa} &= \tfrac{5}{9}\tilde{D}(0)(2\Gamma^{0\lambda}_{\nu\lambda}\Gamma^{0\nu}_{\mu\varkappa}-\Gamma^{0\lambda}_{\varkappa\lambda}\Gamma^{0\nu}_{\mu\nu}-\Gamma^{0\nu}_{\varkappa\lambda}\Gamma^{0\lambda}_{\mu\nu})\\
L_{\mu\varkappa} &= \tfrac{5}{18}\tilde{D}(0)g^{0\nu\lambda}\{-9(\gamma_{\rho;\nu\lambda}\Gamma^{0\rho}_{\mu\varkappa}-\gamma_{\rho;\varkappa\lambda}\Gamma^{0\rho}_{\mu\nu})\\
&\quad +\tfrac{63}{16}(\gamma_{\eta;\nu\lambda}\gamma_{\eta;\mu\varkappa}-\gamma_{\eta;\varkappa\lambda}\gamma_{\eta;\mu\nu})\\
&\quad +\tfrac{1}{2}[g^0_{\eta\eta}(\gamma_{\rho;\nu\lambda}\gamma_{\rho;\mu\varkappa}-\gamma_{\rho;\varkappa\lambda}\gamma_{\rho;\mu\nu})\\
&\quad +\tfrac{1}{2}g^0_{\eta\sigma}(\gamma_{\sigma;\nu\lambda}\gamma_{\eta;\mu\varkappa}-\gamma_{\sigma;\varkappa\lambda}\gamma_{\eta;\mu\nu})]\}\\
M_{\mu\varkappa} &= \tfrac{5}{288}\tilde{D}_1(0)[31g^{0\rho\rho}g^0_{\mu\varkappa}-20g^0_{\mu\varkappa}-14(g^{0\rho\rho})^2\delta_{\mu\varkappa}+40\delta_{\mu\varkappa}]
\end{aligned}
\tag{5.29}
$$

Here the quantities $N^1_{\lambda\mu\lambda\varkappa}$, $\gamma_{\rho;\mu\nu}$, $\tilde{D}(0)$, and $\tilde{D}_1(0)$ are given by the first terms of (5.20), (5.7), (5.26), and (5.27), respectively.

To define the contraction $G_{\mu\nu}R$, we use definition (5.18). After some calculations its averaged value takes the form

$$\begin{aligned}\langle G_{\mu\nu}R\rangle_s = {} & g^0_{\mu\varkappa}R^0 + \tfrac{1}{4}R^0 D^{\rho}_{\mu,\varkappa\rho}(0) - R^0_{\beta\beta'}\langle\varepsilon_{\mu\varkappa}\varepsilon^{\beta\beta'}\rangle \\ & + \tfrac{3}{4}g^0_{\mu\varkappa}R^0_{\beta\beta'}D^{\beta\delta,\beta'}_{\delta}(0) + g^{0\beta\beta'}\langle\varepsilon_{\mu\varkappa}M_{\beta\beta'}\rangle \\ & - g^0_{\mu\varkappa}(g^{0\beta\beta'}\langle Q_{\beta\beta'}\rangle - \langle\varepsilon^{\beta\beta'}M_{\beta\beta'}\rangle)\end{aligned} \tag{5.30}$$

Calculation procedures similar to those carried out above for obtaining the averaged tensor $R_{\mu\varkappa}$, give

$$\begin{aligned}\langle G_{\mu\varkappa}R\rangle = {} & g^0_{\mu\varkappa}R^0 + \tfrac{5}{8}\tilde{D}(0)R^0\delta_{\mu\varkappa} - \tfrac{5}{9}\tilde{D}(0)(2R^0_{\mu\varkappa} - \tfrac{1}{2}\delta_{\mu\varkappa}R^0_{\rho\rho}) \\ & + \tfrac{15}{8}\tilde{D}(0)g^0_{\mu\varkappa}R^0_{\rho\rho} + g^0_{\mu\varkappa}[A + g^{0\beta\beta'}(2N_{\beta\beta'} + M_{\beta\beta'} - 2Q_{\beta\beta'} \\ & + \tfrac{15}{16}A_{\beta\beta'} + L_{\beta\beta'}) - \tfrac{5}{12}\Lambda_{\beta\beta'}] + \Omega_{\mu\varkappa}\end{aligned} \tag{5.31}$$

where

$$\begin{aligned}A = {} & \tfrac{5}{6}\tilde{D}(0)[\tfrac{9}{4}g^{0\beta\beta'}N^1_{\lambda\beta\lambda\beta'} + \tfrac{1}{3}(N^1_{\beta\beta\lambda\lambda} + N^1_{\lambda\beta\beta\lambda} - \tfrac{1}{2}N^1_{\beta\lambda\beta\lambda})] \\ \Omega_{\mu\varkappa} = {} & -\tfrac{5}{18}\tilde{D}(0)g^{0\beta\beta'}(N^1_{\mu\beta\varkappa\beta'} + N^1_{\varkappa\beta\mu\beta'} - \tfrac{1}{2}\delta_{\mu\varkappa}N_{\lambda\beta\lambda\beta'}) \\ & + \tfrac{5}{36}\tilde{D}_1(0)[4g^{0\rho\rho}g^0_{\mu\varkappa} - (g^{0\rho\rho})^2\delta_{\mu\varkappa} - 4g^0_{\mu\varkappa} + 4\delta_{\mu\varkappa}] \\ & + \tfrac{5}{18}\tilde{D}(0)g^{0\beta\beta'}g^{0\nu\lambda}[2\gamma_{\mu;\beta'\lambda}\gamma_{\varkappa;\rho\nu} - \gamma_{\mu;\nu\lambda}\gamma_{\varkappa;\beta\beta'} - \gamma_{\varkappa;\nu\lambda}\gamma_{\mu;\beta\beta'} \\ & - \tfrac{1}{4}\delta_{\mu\varkappa}(\gamma_{\eta;\beta\nu}\gamma_{\eta;\beta'\lambda} - \gamma_{\eta;\beta\beta'}\gamma_{\eta;\nu\lambda})] \\ & + \tfrac{5}{9}\tilde{D}(0)g^{0\beta\beta'}[\tfrac{1}{4}\delta_{\mu\varkappa}\Lambda_{\beta\beta'} - \Gamma^{0\eta}_{\mu\varkappa}\gamma_{\eta;\beta\beta'} \\ & + \tfrac{1}{2}(\Gamma^{0\eta}_{\beta'\mu}\gamma_{\eta;\beta\varkappa} + \Gamma^{0\eta}_{\beta'\varkappa}\gamma_{\eta;\beta\mu}) \\ & + g^{0\nu\lambda}(2\Gamma^0_{\mu;\nu\lambda}\Gamma^0_{\varkappa;\beta\beta'} - \Gamma^0_{\mu;\beta'\lambda}\Gamma^0_{\varkappa;\beta\nu} - \Gamma^0_{\varkappa;\beta'\lambda}\Gamma^0_{\mu;\beta\nu})]\end{aligned}$$

Other quantities in (5.31) are defined by (5.29).

5.4. The Einstein Equation in Space-Time with Stochastic Metric

First we note that it is not difficult to reconstruct the Einstein equation in space-time with a stochastic metric from first principles, as done in the usual theory of gravity. If we use the general covariance principle discussed in Section 3.1, then the corresponding generalization of the Einstein equation may be made by redefining the Ricci tensor $R^0_{\mu\varkappa} \to R_{\mu\varkappa}$, scalar curvature $R^0 \to R$, and the energy-momentum tensor $T^0_{\mu\nu} \to T_{\mu\nu}$, which enter into the usual Einstein equation. In previous sections we defined the Ricci tensor and the scalar curvature R by means of the stochastic metric tensor and averaged their quantities in the limit of weak field $\varepsilon_{\mu\nu}(z)$.

Now the question of how to redefine the *energy-momentum tensor* in space-time with a stochastic metric arises. We assume that its covariant structure is conserved under the transformation $x^\nu \to z^\nu$ of the coordinates. Thus,

$$T_{\mu\nu}(z) = \frac{\partial x^\alpha}{\partial z^\mu}\frac{\partial x^\beta}{\partial z^\nu} T^x_{\alpha\beta} = \frac{\partial x^\alpha}{\partial z^\mu}\frac{\partial x^\beta}{\partial z^\nu}\frac{\partial \xi^\eta}{\partial x^\alpha}\frac{\partial \xi^\delta}{\partial x^\beta} T^f_{\eta\delta} \tag{5.32}$$

where $T^f_{\eta\delta}$ is the energy-momentum tensor in the local-inertial system of reference ξ^α. The energy-momentum tensor in the quasilocal-inertial system of reference x^α with the stochastic metric (1.11) is defined as

$$T^x_{\alpha\beta} = \frac{\partial \xi^\sigma}{\partial x^\alpha}\frac{\partial \xi^\rho}{\partial x^\beta} T^f_{\sigma\rho} \tag{5.33}$$

According to the Jacobian of transformation (1.10), its averaged value is

$$\langle T^x_{\alpha\beta}\rangle_s = T^f_{\alpha\beta}[1 + \tfrac{5}{72}\tilde{D}(0)] + \tfrac{5}{36}\tilde{D}(0)\, T^f_{\lambda\lambda}\delta_{\alpha\beta} \tag{5.34}$$

and therefore, in the presence of an external gravitational field, the energy-momentum averaged tensor takes the form

$$\langle T_{\mu\nu}(z)\rangle_s = T^0_{\mu\nu}[1 + \tfrac{5}{72}\tilde{D}(0)] + \tfrac{5}{36}\tilde{D}(0)\, T^f_{\lambda\lambda}\delta_{\alpha\rho}\frac{\partial x^\alpha}{\partial z^\mu}\frac{\partial x^\rho}{\partial z^\nu} \tag{5.35}$$

where $T^0_{\mu\nu}$ is the usual energy-momentum tensor in the presence of gravity without the stochastic metric, and the connection between coordinates x^ν and z^μ is defined by the standard form as in the usual theory of gravity with the metric $\overset{0}{g}_{\mu\nu}$.

We assume that in the case of a weak static gravitational field generated by a nonrelativistic body with mass density ρ, the 00th component of the stochastic metric tensor is approximately equal to (for example, see Section 2.4)

$$G_{00} \sim -(1 + 2\phi_f)$$

Here ϕ_f is the modified Newtonian potential defined by the *Poisson equation*

$$\nabla^2\phi_f = 4\pi G\rho$$

The energy density T_{00} for a substance moving with a nonrelativistic velocity is proportional to its mass density

$$T_{00} \approx \rho$$

Collecting these two relations, we get

$$\nabla^2 G_{00} = -8\pi G T_{00} \tag{5.36}$$

However, relation (5.36) allows us to assume that the equation for weak fields with energy-momentum distribution $t_{\alpha\beta}$ has the standard form:

$$\lambda_{\alpha\beta} = -8\pi G t_{\alpha\beta}$$

where $\lambda_{\alpha\beta}$ is formed from a linear combination of the metric tensor and its first and second derivatives. Then from the general equivalence principle it follows that the equation which defines a gravitational field with an arbitrary stress in the presence of the stochastic metric must be of the form

$$\Lambda_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{5.37}$$

where $\Lambda_{\mu\nu}$ is a tensor leading to $\lambda_{\alpha\beta}$ in the case of weak fields.

By using the standard method as in the usual theory of gravity, in our case this tensor is given by

$$\Lambda_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{4} G_{\mu\nu} R \tag{5.38}$$

Inserting this into equation (5.37), we get the *Einstein equation* in space-time with the stochastic metric $G_{\mu\nu}$,

$$R_{\mu\nu} - \tfrac{1}{2} G_{\mu\nu} R = -8\pi G T_{\mu\nu} \tag{5.39}$$

The averaging procedure for this equation may be followed using expressions (5.28), (5.31), and (5.35).

5.5. The Bianchi Indentity and the Coordinate Conditions

It turns out that in the gravitational theory with the stochastic metric $G_{\mu\nu}$ the Bianchi identity and the coordinate conditions are fulfilled. The former can be obtained by introducing the quasilocal inertial system of coordinates at the considered point at which $\Gamma^{\lambda}_{\mu\nu}$ is approximately equal to zero up to the order of (l_{Pl}^2/l^2). At the given point, expression (5.13) gives

$$R_{\lambda\mu\nu\varkappa;\eta} = \tfrac{1}{2}\, \partial N_{\lambda\mu\nu\varkappa}/\partial z^{\eta} \tag{5.40}$$

where $N_{\mu\lambda\nu\varkappa}$ is given by (5.12). By cyclic rearrangment of the indices ν, $\varkappa$, and η one can obtain the *Bianchi identity*

$$R_{\lambda\mu\nu\varkappa:\eta} + R_{\lambda\mu\eta\nu;\varkappa} + R_{\lambda\mu\varkappa\eta;\nu} = 0 \tag{5.41}$$

These identities are explicitly covariant, so that they are valid in any system of reference, including quasilocal inertial ones.

The contracted form of (5.41) is sometimes very useful. According to the fact that covariant derivatives of $G^{\lambda\nu}$ disappear, and contracting λ and ν, we find

$$R_{\mu\varkappa;\eta} - R_{\mu\eta;\varkappa} + R^{\nu}_{\mu\varkappa\eta;\nu} = 0 \tag{5.42}$$

Contracting this relation again, one gets

$$R_{;\eta} - R^{\mu}_{\eta;\mu} - R^{\nu}_{\eta;\nu} = 0$$

or

$$(R^{\mu}_{\eta} - \tfrac{1}{2}\delta^{\mu}_{\eta} R)_{;\mu} = 0 \tag{5.43}$$

An equivalent but more well-known formula has the form

$$(R^{\mu\nu} - \tfrac{1}{2}G^{\mu\nu}R)_{;\mu} = 0 \tag{5.44}$$

as in the usual theory of gravity.

To explain the *coordinate conditions* in our scheme with the stochastic metric, we first note that the symmetric tensor $\Lambda_{\mu\nu}$ in (5.38) has ten independent components, and therefore, Einstein's field equations (5.39) consist of ten algebraic independent equations. However, from the Bianchi identity (5.44) it follows that there are not ten functional independent equations, but only $10-4=6$ equations. These equations remain four independent degrees of freedom in ten independent components of the stochastic metric tensor $G_{\mu\nu}$. These degrees of freedom correspond to the fact that if $G_{\mu\nu}$ is a solution of Einstein's equation, then another solution is $G'_{\mu\nu}$ which is obtained from $G_{\mu\nu}$ by means of an arbitrary transformation of coordinates $z^{\nu} \to z'^{\nu}$. This transformation of coordinates gives rise to four arbitrary functions $z'^{\mu}(z)$ corresponding to just four degrees of freedom in the solutions of equation (5.39). Further, by choosing a concrete system of reference, one can eliminate ambiguity in the metric tensor. The choice of the system can be expressed in the form of four coordinate condition, which, by supplementing six independent Einstein equations, lead to a synonymous solution. It is more convenient to use the *condition of harmonicity of coordinates*

$$\Gamma^{\lambda} \equiv G^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0 \tag{5.45}$$

To show that choice of the coordinate system in accordance with these conditions is always possible, recall the transformation equations for the affine connection

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial z'^{\lambda}}{\partial z^{\rho}} \frac{\partial z^{\tau}}{\partial z'^{\mu}} \frac{\partial z^{\sigma}}{\partial z'^{\nu}} \Gamma^{\rho}_{\tau\sigma} = \frac{\partial z^{\rho}}{\partial z'^{\nu}} \frac{\partial z^{\sigma}}{\partial z'^{\mu}} \frac{\partial^2 z'^{\lambda}}{\partial z^{\rho}\, \partial z^{\sigma}}$$

[see equation (3.15)]. Contracting this equation with $G'^{\mu\nu}$, one finds that

$$\Gamma'^{\lambda} = (\partial z'^{\lambda}/\partial z^{\rho})\Gamma^{\rho} - G^{\rho\sigma}\, \partial^2 z'^{\lambda}/\partial z^{\rho}\, \partial z^{\sigma} \tag{5.46}$$

Therefore, if Γ^{ρ} disappears, we can always introduce a new system of coordinates by solving the following second-order partial differential equations:

$$G^{\rho\sigma}\, \partial^2 z'^{\lambda}/\partial z^{\rho}\, \partial z^{\sigma} = (\partial z'^{\lambda}/\partial z^{\rho})\Gamma^{\rho}$$

Then equation (5.46) leads to $\Gamma'^{\lambda}=0$ in the system of coordinates z'^{ν}. Of course, the four conditions (5.46) are not in general covariant, but their necessity is dictated by the elimination of ambiguity which appears in the metric tensor due to the covariance form of Einstein's equation.

Although we cannot write these conditions in the form of covariant equations, we can make them more elegant by expressing the affine connections through the metric tensor:

$$\Gamma^{\lambda}=\tfrac{1}{2}G^{\mu\nu}G^{\lambda\varkappa}\gamma_{\varkappa;\nu\mu}$$

Recall that

$$G^{\lambda\varkappa}\,\partial G_{\varkappa\mu}/\partial z^{\nu}=-G_{\varkappa\mu}\,\partial G^{\lambda\varkappa}/\partial z^{\nu}$$

$$\tfrac{1}{2}G^{\mu\nu}\,\partial G_{\mu\nu}/\partial z^{\varkappa}=G^{-1/2}\,\partial G^{1/2}/\partial z^{\varkappa}$$

[see the formulas (3.39) and (3.41)]. From this it follows that

$$\Gamma^{\lambda}=-G^{-1/2}\,\partial(G^{1/2}G^{\lambda\varkappa})/\partial z^{\varkappa} \tag{5.47}$$

and the conditions leading to *harmonic coordinates* take the form

$$\partial(G^{1/2}G^{\lambda\varkappa})/\partial z^{\varkappa}=0 \tag{5.48}$$

If there exists only the fictitious background radiation field $\varepsilon_{\mu\nu}(x)$, then relation (5.48) becomes the exact equality

$$\partial[(g_{(s)})^{1/2}g_{(s)}^{\alpha\beta}]/\partial x^{\beta}=0 \tag{5.49}$$

Now we explain the term "harmonic coordinates." It signifies that the *function* $F(z)$ is *harmonic* if it satisfies the following equation:

$$\Box F(z)=0$$

where $\Box$ is *D'Alembert's invariant operator*, given by

$$\Box F=(G^{\rho\varkappa}F_{;\rho})_{;\varkappa} \tag{5.50}$$

Making use of (3.36), (3.42), and (3.40), we get

$$\Box F=G^{\lambda\varkappa}\,\partial^2 F/\partial z^{\lambda}\,\partial z^{\varkappa}-\Gamma^{\lambda}\,\partial F/\partial z^{\lambda} \tag{5.51}$$

If $\Gamma^{\lambda}=0$, then the coordinates are harmonic functions (5.50), thus warranting the name "harmonic" for such a system of coordinates.

In the absence of both external gravitational and additional background fields, the explicitly harmonic system of coordinates comprises the Minkowski coordinates in which $G^{\lambda\mu}=\eta^{\lambda\mu}$ and $G=1$, so that the relation (5.48) is fulfilled identically.

6. CONSTRUCTION OF THE THEORY WITH QUANTUM FLUCTUATION OF THE SPACE-TIME METRIC

In this section, instead of additional background radiation stochastic fields $\varepsilon_{\mu\nu}(x)$ considered above, we deal with some *quantized field* $\hat{\varepsilon}_{\mu\nu}(x)$

$$\hat{\varepsilon}_{\mu\nu}(x)=\sum_{\rho}\int d\omega_{\mathbf{k}}[a(\mathbf{k},\rho)e_{\mu\nu}(\mathbf{k},\rho)\,e^{ikx}+a^{+}(\mathbf{k},\rho)e^{*}_{\mu\nu}(\mathbf{k},\rho)\,e^{-ikx}]$$
$$d\omega_{\mathbf{k}}=(2\pi)^{-3/2}(2k^{0})^{-1/2}\,d^{3}k, \qquad k^{0}=|\mathbf{k}| \tag{6.1}$$

where $e_{\mu\nu}(\mathbf{k},\rho)$ is the *polarization tensor* of the graviton with the momentum $\hbar\mathbf{k}$ and the helicity ρ. *Creation* $a^{+}(\mathbf{k},\rho)$ and *annihilation* $a(\mathbf{k},\rho)$ boson operators satisfy the following *commutation rules*:

$$[a(\mathbf{k},\rho),a^{+}(\mathbf{k}',\rho')]=\delta_{\rho\rho'}\delta^{(3)}(\mathbf{k}-\mathbf{k}') \tag{6.2}$$

$$[a(\mathbf{k},\rho),a(\mathbf{k}',\rho')]_{-}=[a^{+}(\mathbf{k},\rho),a^{+}(\mathbf{k}',\rho')]_{-}=0 \tag{6.3}$$

Here our purpose is not to quantize the gravity with the field (6.1), but we use it as a method of introducing *quantum fluctuation in the space-time metric* and consider its consequences in accordance with previous sections.

Notice that with the Hamiltonian constructed by means of (6.1) the quantization of gravity encounters some difficulties caused by the fact that the operator (6.1) cannot be the Lorentz tensor, since the summation over helicities is restricted by physical values of $\rho=\pm 2$, while, as will be shown below, a true tensor would have helicities $0, \pm 1, \pm 2$. From the very beginning we can start from a true tensor, and then subject $e_{\mu\nu}$ to a gradient transformation in order to forbid unphysical values of helicities 0 and ± 1. However, by selecting a gauge in such a way, then $\hat{\varepsilon}_{\mu\nu}(x)$ is already not a tensor. If, instead, we assume that e_{13}, e_{23}, e_{10}, e_{20}, e_{00}, e_{01}, and e_{33} disappear when $\mathbf{k}$ is directed along the third axis, then the gauge condition is not the Lorentz invariant. Indeed, if we make these components equal to zero, then under the Lorentz transformation Λ^{μ}_{ν} the quantity $\hat{\varepsilon}_{\mu\nu}(x)$ does not simply pass to $\Lambda^{\rho}_{\mu}\Lambda^{\sigma}_{\nu}\hat{\varepsilon}_{\rho\sigma}(x)$, but undergoes an additional *gradient transformation*

$$\hat{\varepsilon}_{\mu\nu}(x)\to\Lambda^{\rho}_{\mu}\Lambda^{\sigma}_{\nu}\hat{\varepsilon}_{\rho\sigma}(x)+\partial h_{\mu}/\partial x^{\nu}+\partial h_{\nu}/\partial x^{\mu}$$

where $h_{\mu}(x)$ are arbitrary small functions (see below).

Thus, the construction of the Hamiltonian from the field $\hat{\varepsilon}_{\mu\nu}(x)$ and the derivation of the Lorentz invariant probability transitions represents a more difficult problem [for details, see, e.g., Arnowitt and Deser (1959), Arnowitt *et al.* (1959, 1960, 1061), Dirac (1959), Feynman (1963), Faddeev and Popov (1967), Mandelstam (1968), and DeWitt (1967, 1968)]. Recent achievements in quantum gravity based on new ideas and approaches are extensively given in the proceedings of the second Oxford symposium edited by Isham *et al.* (1981) and of the 11th international conference on general relativity and gravitation edited by MacCallum (1987).

6.1. Weak-Field Approximation

Now we show that the additional term in the space-time metric given by formula (1.11) or (2.13) defines indeed a wavelike solution to the Einstein equation (5.39) for the c-number field $\varepsilon_{\mu\nu}(x)$. To study this problem, we shall turn to the weak-field limit, omitting the square term in the field $\varepsilon_{\mu\nu}(x)$ in (1.11) or (2.13). Thus, we assume that the metric $G_{\mu\nu}$ becomes the Minkowski one,

$$G_{\mu\nu}=\eta_{\mu\nu}+\varepsilon_{\mu\nu} \tag{6.4}$$

where $|\varepsilon_{\mu\nu}(x)|\ll 1$, and therefore we have omitted the term of the type $\varepsilon^{\rho}_{\mu}(x)\varepsilon_{\nu\rho}(x)$; $\varepsilon_{\mu\nu}(x)$ is a c-number field. Further, we follow Weinberg (1972). Thus, in the first order of $\varepsilon_{\mu\nu}(x)$, the *Ricci tensor* has the form

$$R_{\mu\nu}\approx\partial\Gamma^{\lambda}_{\lambda\mu}/\partial x^{\nu}-\partial\Gamma^{\lambda}_{\mu\nu}/\partial x^{\lambda}+O(\varepsilon^{2}) \tag{6.5}$$

and the *affine connection* is

$$\Gamma^{\lambda}_{\mu\nu}=\tfrac{1}{2}\eta^{\lambda\rho}\mathscr{E}_{\rho,\mu\nu}(x) \tag{6.6}$$

where the expression $\mathscr{E}_{\rho,\mu\nu}(x)$ is given by (5.7). When we restrict ourselves to the first order of $\varepsilon_{\mu\nu}$, then the lowering and the raising of all indices should be carried out by means of $\eta^{\mu\nu}$, but not $G^{\mu\nu}$, i.e.,

$$\eta^{\lambda\rho}\varepsilon_{\rho\nu}(x)\equiv\varepsilon^{\lambda}_{\nu}(x),\qquad \eta^{\lambda\rho}\,\partial/\partial x^{\rho}\equiv\partial/\partial x^{\lambda},\qquad \text{and so on}$$

In this approach equations (6.5) and (6.6) give the *Ricci tensor* in the first order:

$$R_{\mu\nu}\approx R^{(1)}_{\mu\nu}=\frac{1}{2}\left(\Box\varepsilon_{\mu\nu}-\frac{\partial^{2}\varepsilon^{\lambda}_{\nu}}{\partial x^{\lambda}\,\partial x^{\mu}}-\frac{\partial^{2}\varepsilon^{\lambda}_{\mu}}{\partial x^{\lambda}\,\partial x^{\nu}}+\frac{\partial^{2}\varepsilon^{\lambda}_{\lambda}}{\partial x^{\mu}\,\partial x^{\nu}}\right) \tag{6.7}$$

Therefore, the *Einstein field equation* is written as

$$\begin{gathered}\Box\varepsilon_{\mu\nu}-\frac{\partial^{2}\varepsilon^{\lambda}_{\nu}}{\partial x^{\lambda}\,\partial x^{\mu}}-\frac{\partial^{2}\varepsilon^{\lambda}_{\mu}}{\partial x^{\lambda}\,\partial x^{\nu}}+\frac{\partial^{2}\varepsilon^{\lambda}_{\lambda}}{\partial x^{\mu}\,\partial x^{\nu}}=-16\pi G S_{\mu\nu}\\ S_{\mu\nu}=T_{\mu\nu}-\tfrac{1}{2}\eta_{\mu\nu}T^{\lambda}_{\lambda}\end{gathered} \tag{6.8}$$

Here $T_{\mu\nu}$ is chosen in the lowest order in $\varepsilon_{\mu\nu}$, and does not depend on $\varepsilon_{\mu\nu}$, satisfying the usual *conservation law*

$$\partial T^{\mu}_{\nu}/\partial x^{\mu}=0 \tag{6.9}$$

Notice that the conservation law (6.9) written in such a form ensures the coordination of equations (6.8), since (6.9) assumes the correctness of

$$\partial S^{\mu}_{\nu}/\partial x^{\mu}=\tfrac{1}{2}\partial S^{\lambda}_{\lambda}/\partial x^{\nu}$$

whereas the linearized *Ricci tensor* satisfies the *Bianchi identity* in the following form:

$$\partial R_{\nu}^{(1)\mu}/\partial x^{\mu}=\tfrac{1}{2}\partial[\Box\varepsilon_{\lambda}^{\lambda}-\partial^{2}\varepsilon^{\lambda\delta}/\partial x^{\lambda}\,\partial x^{\delta}]/\partial x^{\nu}=\tfrac{1}{2}\partial R_{\lambda}^{(1)\lambda}/\partial x^{\nu}$$

As discussed in Section 5.5, a field equation such as (6.8) is not expected to lead to a unique solution, since, given any solution, one can always change coordinates to obtain other solutions. A more general transformation of coordinates, keeping the field to be weak, has the form

$$x^{\mu}\to x'^{\mu}=x^{\mu}+h^{\mu}(x) \tag{6.10}$$

where $\partial h^{\mu}/\partial x^{\nu}$ is of the same order as the $\varepsilon_{\mu\nu}$ field. In the new system of coordinates the metric is written as

$$G'^{\mu\nu}=\frac{\partial x'^{\mu}}{\partial x^{\lambda}}\frac{\partial x'^{\nu}}{\partial x^{\rho}}G^{\lambda\rho}$$

or, since $G^{\mu\nu}\approx\eta^{\mu\nu}-\varepsilon^{\mu\nu}$, one can write

$$\varepsilon'^{\mu\nu}=\varepsilon^{\mu\nu}-\eta^{\lambda\nu}\,\partial h^{\mu}/\partial x^{\lambda}-\eta^{\rho\mu}\,\partial h^{\nu}/\partial x^{\rho}$$

Thus, if $\varepsilon_{\mu\nu}$ is a solution of equation (6.8), then so should

$$\varepsilon'_{\mu\nu}=\varepsilon_{\mu\nu}-\partial h_{\mu}/\partial x^{\nu}-\partial h_{\nu}/\partial x^{\mu} \tag{6.11}$$

where $h_{\mu}\equiv h^{\nu}\eta_{\mu\nu}$ are four small quantities and, generally speaking, arbitrary functions of x^{ν}. Substituting (6.11) into (6.8), it is easy to verify immediately that (6.11) is also its solution. This property is a consequence of the so-called *gauge invariance* of the field equation.

The gauge invariance of the field equation (6.8) gives rise to difficulties when we want to solve it exactly. However, these difficulties can be removed by choosing particular gauge, i.e., some system of coordinates. It is more convenient to work in the *harmonic system of reference* for which

$$G^{\mu\nu}\Gamma_{\mu\nu}^{\lambda}=0$$

By using (6.6), then in the first order one gets

$$\partial\varepsilon_{\nu}^{\mu}/\partial x^{\mu}=\tfrac{1}{2}\,\partial\varepsilon_{\mu}^{\mu}/\partial x^{\nu} \tag{6.12}$$

Such a choice is always possible, and follows from the general arguments expounded in Section 5.5. From the expression (6.11) it is also seen that if $\varepsilon_{\mu\nu}$ does not satisfy the condition (6.11), then by carrying out some transformation of the coordinates (6.10) provided

$$\Box h_{\nu}=\partial\varepsilon_{\nu}^{\mu}/\partial x^{\mu}-\tfrac{1}{2}\,\partial\varepsilon_{\mu}^{\mu}/\partial x^{\nu}$$

we find some tensor $\varepsilon'_{\mu\nu}$ which has already satisfied the condition (6.12).

Therefore, it will be assumed that $\varepsilon_{\mu\nu}$ is indeed the solution of equation (6.12). Inserting (6.12) into (6.8), we can write the field equation in the form

$$\Box \varepsilon_{\mu\nu} = -4\pi G S_{\mu\nu} \tag{6.13}$$

One of the solutions represents the *retarded potential*

$$\varepsilon_{\mu\nu}(\mathbf{x}, t) = 4\pi \int d^3x' \, |\mathbf{x}-\mathbf{x}'|^{-1} S_{\mu\nu}(\mathbf{x}', t-|\mathbf{x}-\mathbf{x}'|) \tag{6.14}$$

As mentioned above, the *conservation law* (6.9) for $T^{\mu\nu}$ is equivalent to

$$\partial S^{\mu}_{\nu}/\partial x^{\mu} = \tfrac{1}{2}\partial S^{\mu}_{\mu}/\partial x^{\nu} \tag{6.15}$$

and in consequence of this, the solution (6.14) for the source $S_{\mu\nu}$ located in a finite volume automatically satisfies the harmonic coordinate condition (6.12). To the solution (6.14) can be added any solution of the homogeneous equation

$$\Box \varepsilon_{\mu\nu} = 0 \tag{6.16}$$

$$\partial \varepsilon^{\mu}_{\nu}/\partial x^{\mu} = \tfrac{1}{2}\, \partial \varepsilon^{\mu}_{\mu}/\partial x^{\nu} \tag{6.17}$$

We understand the expression (6.14) as the gravitational radiation generated by the source $S_{\mu\nu}$, whereas any additional term satisfying (6.16) and (6.17) represents gravitational radiation coming from infinity. The appearance of the time variable $t-|\mathbf{x}-\mathbf{x}'|$ in (6.14) shows that gravitational effects propagate with the single velocity ($c=1$), i.e., with the velocity of the light.

6.2. Plane Wave Solutions

Let us consider the plane wave solutions of the homogeneous equations (6.16) and (6.17), since they play an important role in understanding the physical nature of the gravitational radiation field and, moreover, as shown below, retarded waves become plane ones at $r \to \infty$. A general solution to equations (6.16) and (6.17) is a linear superposition of solutions, written in the form

$$\varepsilon_{\mu\nu}(x) = e_{\mu\nu}\, e^{ikx} + e^{*}_{\mu\nu}\, e^{-ikx} \tag{6.18}$$

Such a solution satisfies equation (6.16) if

$$k_{\mu} k^{\mu} = 0 \tag{6.19}$$

and the condition (6.17) if the relation

$$k_{\mu}\, e^{\mu}_{\nu} = \tfrac{1}{2} k_{\nu}\, e^{\mu}_{\mu} \tag{6.20}$$

holds. It is obvious that the matrix $e_{\mu\nu}$ is symmetric:

$$e_{\mu\nu} = e_{\nu\mu} \tag{6.21}$$

We will call it the *polarization tensor.*

In the general case, a symmetric matrix (4×4) has ten independent components. However, the four relations (6.20) reduce their number to six and of these six components only two have the meaning of degree of physical freedom. By carrying out the transformation of coordinates $x^\mu \to x^\mu + h^\mu(x)$, we change the metric $\eta_{\mu\nu}+\varepsilon_{\mu\nu}$ to the new metric $\eta_{\mu\nu}+\varepsilon'_{\mu\nu}$, where $\varepsilon'_{\mu\nu}$ is given by the expression (6.11). Assume that we choose $h^\mu(x)$ in the form

$$h^\mu(x) = ih^\mu\, e^{ikx} - ih^{*\mu}\, e^{-ikx} \tag{6.22}$$

Then (6.11) leads to the expression

$$e'_{\mu\nu}(x) = e'_{\mu\nu}\, e^{ikx} + e'^{*}_{\mu\nu}\, e^{-ikx} \tag{6.23}$$

where

$$e'_{\mu\nu} = e_{\mu\nu} + k_\mu h_\nu + k_\nu h_\mu \tag{6.24}$$

Notice that waves nevertheless satisfy the harmonic coordinate condition (6.20). It may be concluded that for four arbitrary parameters h_μ, the polarization tensors $e'_{\mu\nu}$ and $e_{\mu\nu}$ correspond to the same physical picture. Namely, from six independent components satisfying (6.20) and (6.21) only $6-4=2$ have physical meaning. For example, consider a wave with wave vector

$$k^1 = k^2 = 0, \qquad k^3 = k^0 \equiv k > 0 \tag{6.25}$$

propagating along the z axis in the direction of increasing values of z. In this case, the relation (6.20) reduces to the conditions

$$e_{31} + e_{01} = e_{32} + e_{02} = 0$$

$$e_{33} + e_{03} = -e_{03} - e_{00} = \tfrac{1}{2}(e_{11} + e_{22} + e_{33} - e_{00})$$

These four relations allow us to express e_{i0} and e_{22} through the other six components $e_{\mu\nu}$,

$$e_{01} = -e_{31}, \qquad e_{02} = -e_{32}, \qquad e_{03} = -\tfrac{1}{2}(e_{33} + e_{00}), \qquad e_{22} = -e_{11} \tag{6.26}$$

Then in the system of coordinates transformed by the formulas (6.10) and (6.22), these six independent components $e_{\mu\nu}$ are changed in accordance with equations (6.24) by the components $e'_{\mu\nu}$:

$$e'_{11} = e_{11} \qquad e'_{12} = e_{12}$$

$$e'_{13} = e_{13} + kh_1 \qquad e'_{23} = e_{23} + kh_2$$

$$e'_{33} = e_{33} + 2kh_1 \qquad e'_{00} = e_{00} - 2kh_0$$

Only the components e_{11} and e_{22} have a true physical meaning. Indeed, one can always find a transformation of coordinates with

$$h_1 = -e_{13}/k, \qquad h_2 = -e_{23}/k, \qquad h_3 = -e_{33}/2k, \qquad h_0 = e_{00}/2k$$

which renders all components of $e'_{\mu\nu}$ zero with the exception of e'_{11}, e'_{12}, and $e'_{22}=-e'_{11}$.

The difference between the separate components of the polarization tensor becomes clear if we understand them by the changing of $\varepsilon_{\mu\nu}(x)$ upon rotation of the system of coordinates around the z axis, i.e., under the following *Lorentz transformation*:

$$\begin{array}{ll} \Lambda_1^1=\cos\theta & \Lambda_1^2=\sin\theta \\ \Lambda_2^1=-\sin\theta & \Lambda_2^2=\cos\theta \\ \Lambda_3^3=\Lambda_0^0=1 & \text{all other } \Lambda_\nu^\mu=0 \end{array} \tag{6.27}$$

Since such a transformation assumes the vector k_μ to be invariant ($\Lambda_\mu^\nu k_\nu = k_\mu$), then only the *polarization tensor* is subjected to the transformation

$$e'_{\mu\nu}=\Lambda_\mu^\rho\Lambda_\nu^\sigma\, e_{\rho\sigma} \tag{6.28}$$

Making use of relation (6.26), we find

$$e'_\pm=\exp(\pm 2i\theta)e_\pm, \qquad f'_\pm=\exp(\pm 2i\theta)f_\pm, \qquad e'_{33}=e_{33}, \qquad e'_{00}=e_{00} \tag{6.29}$$

where

$$\begin{aligned} e_\pm &\equiv e_{11}\mp ie_{12}=-e_{22}\mp ie_{12} \\ f_\pm &\equiv e_{31}\mp ie_{32}=-e_{01}\pm ie_{02} \end{aligned} \tag{6.30}$$

Let us say that any plane wave ψ transforming by the rule

$$\psi'=e^{ih\theta}\psi \tag{6.31}$$

under rotation of the angle θ with respect to the direction of the spreading of the wave has helicity h. So, it is seen that the gravitational plane wave can be decomposed into the following components: $e_\pm$ possessing helicity ± 2; $f_\pm$ with helicity ± 1; and also e_{00} and e_{33} with zero helicity. However, it is easily proved that the components with helicities 0 and ± 1 become zero by the appropriate choice of the system of coordinates, and therefore, only the components with helicity ± 2 have physical meaning.

It is useful to compare the above formalism with electrodynamics. The *Maxwell equations* in the *Lorentz gauge* have the form

$$\partial^\alpha A_\alpha=0, \qquad \Box A_\alpha=-J_\alpha \tag{6.32}$$

In empty space these equations acquire the analogous form of (6.16) and (6.17):

$$\Box A_\alpha=0, \qquad \partial A^\alpha/\partial x^\alpha=0$$

for the metric written in harmonic coordinates. Here we deal with the inertial system of coordinates, and therefore,

$$\Box = \eta^{\alpha\beta}\, \partial^2/\partial x^\alpha\, \partial x^\beta$$

The solution of the given equations, as for equations (6.18)–(6.20), can be written in the form of the plane wave

$$A_\alpha = e_\alpha\, e^{ikx} + e_\alpha^*\, e^{-ikx}$$

where

$$k^\alpha k_\alpha = 0, \qquad k_\alpha\, e^\alpha = 0$$

Generally speaking, e^α would have four independent components, but the condition $k_\alpha\, e^\alpha = 0$ reduces the number of independent components to three, while the condition (6.20) increases the number of independent components $e_{\mu\nu}$ to six. Further, with the Lorentz gauge and the unchanging physical fields **E** and **B**, analogously with (6.11) and (6.22), one can change A_α by using the *gauge transformation*

$$A_\alpha \to A'_\alpha = A_\alpha + \partial f/\partial x_\alpha, \qquad f(x) = i\varepsilon\, e^{ikx} - i\varepsilon^*\, e^{-ikx}$$

By analogy with (6.23) and (6.24) a new potential can also be written in the form

$$A'_\alpha = e'_\alpha\, e^{ikx} + e'^*_\alpha\, e^{-ikx}, \qquad e'_\alpha = e_\alpha - \delta \cdot k_\alpha$$

The parameter δ is arbitrary, so that, of the three algebraically independent components e_α, only two have physical meaning, just as the general covariance leaves physical meaning for only two of the six independent components. In order to isolate these two components e_α, consider a wave propagating along the z axis with the vector k^α given by the relations (6.25). Then from the condition $k_\alpha\, e^\alpha = 0$ there follows the equality $e_0 = -e_3$, whereas the condition (6.20) allows us to express e_{22} and e_{0i} through another six components $e_{\mu\nu}$. Further, the considered gauge transformation leaves e_1 and e_2 invariant, but changes e_3 by

$$e'_3 = e_3 - \delta \cdot k$$

Therefore, choosing $\delta = e_3/k$, one can render e'_3 zero and as a result only e_1 and e_2 possess physical meaning, while e_{11} and e_{22} alone do not become zero by any transformation of coordinates. Finally, the physical meaning of the given two components can be found by subjecting the electromagnetic plane wave to rotation (6.27). The polarization vector is changed by

$$e'_\alpha = \Lambda_\alpha^\beta e_\beta$$

and therefore,

$$e'_\pm = \exp(\pm i\theta) e_\pm, \qquad e'_3 = e_3$$

where

$$e_{\pm} = e_1 \mp ie_2$$

Thus, electromagnetic waves are decomposed into components with helicity ±1 and 0. However, physical meaning belongs only to components with helicity ±1, but not 0, just as gravitational waves may have helicity ±2, but not ±1 or 0. All these considerations are valid when we use classical language and say that electromagnetic and gravitational perturbations are carried out over waves with spin 1 and 2, respectively.

6.3. Quantization of the Metric Tensor

We see that with the quantum field (6.1) the metric tensor (1.11) in the absence of an external gravitational field now takes the quantized form

$$\hat{g}_{\mu\nu}(x) = \eta_{\mu\nu} + \hat{\varepsilon}_{\mu\nu}(x) + \tfrac{1}{4}\hat{\varepsilon}^{\alpha}_{\mu}(x)\hat{\varepsilon}_{\nu\alpha}(x) \tag{6.33}$$

Theorem 6.1. Let $\hat{\varepsilon}_{\mu\nu}(x)$ be the quantized field (6.1) satisfying the *commutation relation*

$$[\hat{\varepsilon}_{\mu\nu}(x), \hat{\varepsilon}_{\rho\sigma}(y)] = iD_{\mu\nu,\rho\sigma}(x-y) \tag{6.34}$$

where $D_{\mu\nu,\rho\sigma}(x)$ is a Pauli–Jordan-like function of the graviton field. Then the commutation rule

$$\langle 0|[\hat{g}_{\mu\nu}(x), \hat{g}_{\rho\sigma}(y)]_{-}|0\rangle = iD_{\mu\nu,\rho\sigma}(x-y) + I_{\mu\nu,\rho\sigma}(x-y) \tag{6.35}$$

holds for the operator-valued metric tensor $\hat{g}_{\mu\nu}(x)$, where the symbol $|0\rangle$ denotes the *vacuum state*:

$$a|0\rangle = 0, \qquad \langle 0|a^{+} = 0, \qquad \langle 0|0\rangle = 1 \tag{6.36}$$

and

$$\begin{aligned} I_{\mu\nu,\rho\sigma}&(x-y) \\ &= \sum_{\rho_1,\rho_2} \iint d\omega_{\mathbf{k}_1}\, d\omega_{\mathbf{k}_2}\, (2\pi)^{-3}(k_1^0 k_2^0)^{-1/2} \\ &\times \{e^{i(k_1+k_2)(x-y)}[e^{\alpha}_{\mu}(\mathbf{k}_1,\rho_1)e_{\nu\alpha}(\mathbf{k}_2,\rho_2)e^{*\varkappa}_{\rho}(\mathbf{k}_2,\rho_2)e^{*}_{\delta\varkappa}(\mathbf{k}_1,\rho_1) \\ &+ e^{\alpha}_{\mu}(\mathbf{k}_1,\rho_1)e_{\nu\alpha}(\mathbf{k}_2,\rho_2)e^{*\varkappa}_{\rho}(\mathbf{k}_1,\rho_1)e^{*}_{\delta\varkappa}(\mathbf{k}_2,\rho_2)] - \text{h.c.}\} \end{aligned} \tag{6.37}$$

Proof. Direct calculation shows that owing to the commutator (6.2) in the expression

$$\begin{aligned} [\hat{g}_{\mu\nu}(x), \hat{g}_{\rho\sigma}(y)]_{-} = [\hat{\varepsilon}_{\mu\nu}(x), \hat{\varepsilon}_{\rho\sigma}(y)]_{-} &+ \tfrac{1}{4}[\hat{\varepsilon}^{\delta}_{\mu}(x)\hat{\varepsilon}_{\nu\delta}(x), \varepsilon_{\rho\sigma}(y)]_{-} \\ &+ \tfrac{1}{4}[\hat{\varepsilon}_{\mu\nu}(x), \hat{\varepsilon}^{\varkappa}_{\rho}(y)\hat{\varepsilon}_{\delta\varkappa}(y)]_{-} \\ &+ \tfrac{1}{16}[\hat{\varepsilon}^{\delta}_{\mu}(x)\hat{\varepsilon}_{\nu\delta}(x), \hat{\varepsilon}^{\varkappa}_{\rho}(y)\hat{\varepsilon}_{\sigma\varkappa}(y)]_{-} \end{aligned}$$

terms of the type a^+, a, and a^+a appear, and, after taking the expectation value over the vacuum state (6.36), it acquires the form (6.35), i.e., the commutator (6.35) is a c-number and depends on a function of the difference $x-y$ of coordinates.

Due to the explicit form of (6.33), the metric tensor in the case of a quantum fluctuation of the space-time metric possesses the *antisymmetric property*

$$\hat{g}_{\mu\nu}(x)-\hat{g}_{\nu\mu}(x)=\tfrac{1}{4}[\hat{\varepsilon}^{\alpha}_{\mu}(x),\hat{\varepsilon}_{\nu\alpha}(x)]=i\tfrac{1}{4}D^{\alpha}_{\mu,\nu\alpha}(0)$$

Further, we use only the symmetric metric tensor defined by means of the *T-product of operators* $\hat{\varepsilon}_{\mu\nu}(x)$:

$$\hat{g}_{\nu\mu}(x)\equiv\hat{g}_{\mu\nu}(x)=\eta_{\mu\nu}+\hat{\varepsilon}_{\mu\nu}(x)+\tfrac{1}{4}T(\hat{\varepsilon}^{\alpha}_{\mu}(x)\varepsilon_{\nu\alpha}(x)) \tag{6.38}$$

and therefore,

$$\langle 0|\hat{g}_{\mu\nu}(x)|0\rangle=\eta_{\mu\nu}+\tfrac{1}{4}D^{c\alpha}_{\mu,\nu\alpha}(0) \tag{6.39}$$

where, by definition,

$$D^{c}_{\mu\nu,\rho\sigma}(x)=i^{-1}(2\pi)^{-4}\int d^4p\, e^{-ipx}\Pi_{\mu\nu,\rho\sigma}(p)(p^2-i\varepsilon)^{-1} \tag{6.40}$$

is the *Green function* of the graviton field. Here the *projecting tensor* $\Pi_{\mu\nu,\rho\sigma}(p)$ for the spin-two field possesses the properties (1.16). Moreover, one can easily verify that

$$\begin{aligned}\langle 0|T[\hat{g}_{\mu\nu}(x)\hat{g}_{\rho\delta}(y)]|0\rangle=&\ \eta_{\rho\delta}\eta_{\mu\nu}+\tfrac{1}{4}\eta_{\rho\delta}D^{c\varkappa}_{\mu,\nu\varkappa}(0)\\&+D^{c}_{\mu\nu,\rho\delta}(x-y)+\tfrac{1}{4}\eta_{\mu\nu}D^{c\varkappa}_{\rho,\delta\varkappa}(0)\\&+\tfrac{1}{16}[D^{c\gamma}_{\mu,\nu\gamma}(0)D^{c\varkappa}_{\rho,\delta\varkappa}(0)+D^{c\gamma,\varkappa}_{\mu,\rho}(x-y)D^{c}_{\nu\gamma,\delta\varkappa}(x-y)\\&+D^{c\gamma}_{\mu,\delta\varkappa}(x-y)D^{c,\varkappa}_{\nu\gamma,\rho}(x-y)]\end{aligned} \tag{6.41}$$

In accordance with the definition (1.12), the inverse metric tensor (1.13a) with respect to the tensor (6.33) in the given case reads

$$\hat{g}^{\nu\sigma}(x)=\eta^{\alpha\beta}\delta^{\nu}_{\rho}\delta^{\sigma}_{\varkappa}T\left\{\int_0^{\infty}d\beta\,\exp[-\beta(\delta^{\alpha}_{\rho}+\tfrac{1}{2}\hat{\varepsilon}^{\alpha}_{\rho}(x))(\delta^{\beta}_{\varkappa}+\tfrac{1}{2}\hat{\varepsilon}^{\beta}_{\varkappa}(x))]\right\}$$

or in the *weak quantized field limit*

$$\hat{g}^{\nu\sigma}(x)=\eta^{\nu\sigma}-\hat{\varepsilon}^{\nu\sigma}(x)+\tfrac{3}{4}T\{\hat{\varepsilon}^{\nu\rho}(x)\hat{\varepsilon}^{\sigma}_{\rho}(x)\} \tag{6.42}$$

Thus, introduction of a quantum fluctuation in the metric allows us to quantize the metric tensor and establish the geometric properties of space-time with a graviton field. ■

6.4. Dyadic Representation

In our scheme, the basic tensor $D^c_{\mu\nu,\rho\sigma}(x)$ is formed by means of projected matrices $\Pi_{\mu\nu,\rho\sigma}(k)$ for which the *dyadic representation*

$$\Pi^{\mu\nu,\rho\sigma}(p) = \sum_{\rho} e^{\mu\nu}(p,\rho)\, e^{\rho\sigma *}(p,\rho) \tag{6.43}$$

holds, where five symmetric tensors $e^{\mu\nu}(p,\rho)$ satisfy the conditions

$$p_\mu e^{\mu\nu}(p,\rho) = 0, \qquad \eta_{\mu\nu} e^{\mu\nu}(p,\rho) = 0, \qquad e^{*\mu\nu}(p,\rho) e_{\mu\nu}(p,\rho') = \delta_{\rho\rho'} \tag{6.44}$$

If one can use helicitic states

$$d^{\mu\nu}(p) = \sum_{\rho=-1}^{+1} e^{\mu}(p,\rho)\, e^{*\nu}(p,\rho), \qquad e^{*\nu}(p,\rho) = (-1)^{\rho} e^{\nu}(p,-\rho) \tag{6.45}$$

in dyads constructed from vectors, then we obtain

$$\Pi^{\mu\nu,\rho\sigma}(p) = \sum_{\rho,\rho'} e^{\mu\nu}(p,\rho,\rho')\, e^{\rho\sigma}(p,\rho,\rho') \tag{6.46}$$

where the tensors entering into the right-hand side of this expression are written through helicitic states with $\rho = \pm 1$ as follows:

$$e^{\mu\nu}(p,\rho,\rho') = \frac{1}{2}\Big[e^{\mu}(p,\rho)\, e^{\nu}(p,\rho') + e^{\mu}(p,\rho')\, e^{\nu}(p,\rho) - \delta_{-\rho\rho'} \sum_{\rho_1} e^{\mu}(p,\rho_1)\, e^{\nu}(p,-\rho_1) \Big] \tag{6.47}$$

Appearing in the last formula are three independent tensors having the form

$$e^{\mu\nu}(p,+1,-1) = 0$$

and

$$e^{\mu\nu}(p,\pm 2) = e^{\mu\nu}(p,\pm 1,\pm 1) = e^{\mu}(p,\pm 1)\, e^{\nu}(p,\pm 1)$$

and therefore, they describe two *helicitic states* of the graviton.

It is useful to notice that the metric tensor $\eta_{\mu\nu}$ can also be formed by the dyadic representation for the photon field. Since the vector p^{μ} is isotropic, $p^2 = 0$. Let us introduce the vector $\bar{p}^{\mu}$ obtained from the vector p^{μ} by conversion of the direction of the photon motion:

$$\bar{p}^0 = p^0, \qquad \bar{p}^i = -p^i, \qquad \bar{p}^2 = 0$$

Then $p^{\mu} + \bar{p}^{\mu}$ will be a timelike vector, but $p^{\mu} - \bar{p}^{\mu}$ a spacelike vector. One can add to them two orthodiagonal unitary spacelike vectors $e^{\mu}(p,\rho)$:

$$e^{\mu *}(p,\rho) e_{\mu}(p,\rho') = \delta_{\rho\rho'}, \qquad p_\mu e^{\mu}(p,\rho) = 0, \qquad \bar{p}_\mu e^{\mu}(p,\rho) = 0$$

and obtain the following *dyadic representation*:

$$\eta^{\mu\nu} = (2p\bar{p})^{-1}(p^\mu + \bar{p}^\mu)(p^\nu + \bar{p}^\nu) - (2p\bar{p})^{-1}(p^\mu - \bar{p}^\mu)(p^\nu - \bar{p}^\nu)$$
$$+ \sum_\rho e^\mu(p, \rho)\, e^{\nu *}(p, \rho)$$
$$= (p\bar{p})^{-1}[p^\mu \bar{p}^\nu + p^\nu \bar{p}^\mu] + \sum_\rho e^\mu(p, \rho)\, e^{\nu *}(p, \rho) \tag{6.48}$$

Necessity in the choice of complex vectors appears when we want to study the angular momentum of a particle. Under the *infinitesimal homogeneous Lorentz transformation*

$$x'^\mu = x^\mu + \delta\omega^{\mu\nu} x_\nu$$

the *current vector* $J^\nu(x)$ is changed according to the law

$$J'^\mu(x') = J^\mu(x) + \delta\omega^{\mu\nu} J_\nu(x)$$

or

$$\delta J^\lambda(x) = \delta\omega^{\mu\nu} x_\mu\, \partial_\nu J^\lambda(x) + \delta\omega^{\lambda\nu} J_\nu(x)$$

from which one can obtain

$$\delta \mathbf{J}(x) = \{\delta\boldsymbol{\omega} \cdot [\mathbf{x} \times \boldsymbol{\nabla}]\}\mathbf{J}(x) - \delta\boldsymbol{\omega} \times \mathbf{J}(x)$$
$$\delta J^0(x) = \delta\boldsymbol{\omega} \cdot [\mathbf{x} \times \boldsymbol{\nabla}] J^0(x)$$

for the three-dimensional rotation, or in the equivalent form,

$$\delta \mathbf{J}(p) = \{\delta\boldsymbol{\omega} \cdot [\mathbf{p} \times \partial/\partial\mathbf{p}]\}\mathbf{J}(p) - \delta\boldsymbol{\omega} \times \mathbf{J}(p)$$
$$\delta J^0(p) = \delta\boldsymbol{\omega} \cdot [\mathbf{p} \times \partial/\partial\mathbf{p}] J^0(p)$$

Let us now consider rotation along the axis directed to the momentum of the particle, when

$$\delta\boldsymbol{\omega} = \delta\varphi\, \mathbf{p}/|\mathbf{p}|$$

A single-particle state with the helicity ρ for which

$$\delta J(p, \rho) = i\rho\, \delta\varphi\, J(p, \rho)$$

is given if and only if

$$-\mathbf{e}^*(p, \rho) \times \mathbf{p}/|\mathbf{p}| = i\rho \mathbf{e}^*(p, \rho)$$

The vector **e** which is parallel to **p** corresponds to zero helicity, and therefore we introduce a new notation,

$$\mathbf{e}(p, 0) = (p^0/m)\mathbf{p}/|\mathbf{p}|, \qquad e^0(p, 0) = |\mathbf{p}|/m$$

for the vector $e^{\mu}(p, 3)$. *Helicitic states* with $\rho = \pm 1$ correspond to the complex combinations

$$\mathbf{e}^*(p, +1) = \frac{1}{\sqrt{2}}[-\mathbf{e}(p, 1) + i\mathbf{e}(p, 2)]$$
$$\mathbf{e}(p, -1) = \frac{1}{\sqrt{2}}[\mathbf{e}(p, 1) + i\mathbf{e}(p, 2)] \tag{6.49}$$

which are chosen in such a way that the relation

$$-\mathbf{e}^*(p, \rho) \times \delta\boldsymbol{\omega} = i \sum_{\rho'=-\rho}^{+1} (\delta\boldsymbol{\omega} \cdot \mathbf{S})_{\rho\rho'} \mathbf{e}^*(p, \rho')$$

leads to the standard matrix elements of the operator of the unitary spin.

The *dyadic representation* is useful for expressing geometric relations of the type of (6.35) and (6.41) by means of the projecting tensor $\Pi_{\mu\nu,\rho\sigma}(k)$ or in the next step by the Green functions $D_{\mu\nu,\rho\sigma}(x)$ and $D^c_{\mu\nu,\rho\sigma}(x)$. For example, the function (6.35) now takes the form

$$I_{\mu\nu,\rho\sigma}(x-y) = \iint d\omega_{\mathbf{k}_1}\, d\omega_{\mathbf{k}_2}\, (2\pi)^{-3}(2k_1^0 2k_2^0)^{-1/2}$$
$$\times \{e^{i(k_1+k_2)(x-y)}[\Pi^{\alpha,}_{\mu,\delta\varkappa}(k_1)\Pi^{\varkappa}_{\nu\alpha,\rho}(k_2)$$
$$+\Pi^{\alpha,\varkappa}_{\mu,\rho}(k_1)\Pi_{\nu\alpha,\delta\varkappa}(k_2)] - \text{h.c.}\}$$

6.5. Green Functions of the Field $\hat{\varepsilon}_{\mu\nu}(x)$

The dyadic representation is convenient for direct calculation of the *Green functions* of the graviton field $\hat{\varepsilon}_{\mu\nu}(x)$. First, let us consider the commutator

$$iD_{\mu\nu,\rho\sigma}(x-y) = [\hat{\varepsilon}_{\mu\nu}(x), \hat{\varepsilon}_{\rho\sigma}(y)]_- \tag{6.50}$$

Substituting the representation (6.1) into (6.50) and using (6.2) and (6.3), we obtain

$$D_{\mu\nu,\rho\sigma}(x-y) = i^{-1}(2\pi)^{-3} \int \frac{d^3k}{2k^0} \Pi_{\mu\nu,\rho\sigma}(k)[e^{ik(x-y)} - e^{-ik(x-y)}] \tag{6.51}$$

where the *projecting tensor* $\Pi_{\mu\nu,\rho\sigma}(k)$,

$$\Pi_{\mu\nu,\rho\sigma}(k) = d_{\mu\rho}(k)d_{\nu\sigma}(k) + d_{\mu\sigma}(k)d_{\nu\rho}(k) - \tfrac{2}{3}d_{\mu\nu}(k)d_{\rho\sigma}(k)$$
$$d_{\mu\nu}(k) = \eta_{\mu\nu} - k_\mu k_\nu / k^2 \tag{6.52}$$

is given by the *dyadic representation*

$$\Pi_{\mu\nu,\rho\sigma}(k) = \sum_\rho e_{\mu\nu}(k, \rho)e^*_{\rho\sigma}(k, \rho)$$

To present the tensor (6.52) by means of the differential operator in x space, we introduce the operator-valued *divisor*

$$\hat{d}_{\mu\nu}(\Box_x, \partial_x^2) = (-\eta_{\mu\nu}\Box + \partial^2/\partial x^\nu \partial x^\mu)$$
$$\Box = \eta^{\alpha\beta}\, \partial^2/\partial x^\alpha\, \partial x^\beta, \qquad \partial_x^2 = \partial^2/\partial x^\nu\, \partial x^\mu \tag{6.53}$$

and use the identity

$$\hat{d}_{\mu\nu}\, e^{ik(x-y)} = d'_{\mu\nu}(k)\, e^{ik(x-y)}, \qquad d'_{\mu\nu}(k) = k^2\eta_{\mu\nu} - k_\mu k_\nu$$

It is easy to verify that the formal differential operator

$$\hat{\Pi}_{\mu\nu,\rho\sigma}(\Box, \partial_{xy}^2) = \int_0^\infty d\alpha\, e^{-\alpha\Box^2}\Pi'_{\mu\nu,\rho\sigma}(\Box, \partial_{xy}^2) \tag{6.54}$$

gives the projecting tensor (6.52) in the momentum space. Here $\Pi'_{\mu\nu,\rho\sigma}(\Box, \partial_{xy}^2)$ is formed through the operator (6.53) in accordance with the formula (6.52), where (6.53) should be put instead of $d_{\mu\nu}(k)$. Notice that the operator (6.54) has physical meaning only in the momentum space. Thus, the function (6.51) can be rewritten in the form

$$D_{\mu\nu,\rho\sigma}(x-y) = \Pi_{\mu\nu,\rho\sigma}(\Box, \partial_{xy}^2)D_0(x-y) \tag{6.55}$$

where

$$D_0(x-y) = i^{-1}(2\pi)^{-3}\int \frac{d^3k}{2k^0}[e^{ik(x-y)} - e^{-ik(x-y)}]$$
$$= i^{-1}(2\pi)^{-3}\int d^4k\, \varepsilon(k^0)\delta(k^2)\, e^{ik(x-y)} \tag{6.56}$$

is the *Pauli–Jordan commutator function* of a scalar particle with mass $m = 0$. Calculation of the explicit form of the different two-point *Green functions* for particles with spin 0, 1, and 1/2 is carried out in textbooks of field theory (see, for example, Bogolubov and Shirkov, 1980). We use here their results. Thus, we have

$$D_0(x) = \frac{1}{2\pi}\,\varepsilon(x^0)\delta(\lambda), \qquad \lambda = -x_0^2 + \mathbf{x}^2 \tag{6.57}$$

The well-known discontinuous function $\varepsilon(x^0)$ entering into expressions (6.56) and (6.57) is given by

$$\varepsilon(x^0) = \theta(x^0) - \theta(-x^0) = \begin{cases} 1 & \text{for } x^0 > 0 \\ -1 & \text{for } x^0 < 0 \end{cases}$$

where $\theta(x^0)$ is the *Heaviside function.*

Now let us introduce functions $D^{(\pm)}_{\mu\nu,\rho\sigma}(x)$ according to

$$D^{(-)}_{\mu\nu,\rho\sigma}(x-y)=D^{(+)}_{\mu\nu,\rho\sigma}(y-x)=i\langle 0|\hat{\varepsilon}_{\mu\nu}(x)\hat{\varepsilon}_{\rho\sigma}(y)|0\rangle$$

Thus, we have

$$\begin{aligned}D^{(-)}_{\mu\nu,\lambda\sigma}(x-y)&=i(2\pi)^{-3}\int\frac{d^3k}{2k^0}\,e^{ik(x-y)}\sum_{\rho}e_{\mu\nu}(k,\rho)e^{*}_{\lambda\sigma}(k,\rho)\\&=\hat{\Pi}_{\mu\nu,\lambda\sigma}(\Box,\partial^2_{xy})D^{(-)}_0(x-y)\end{aligned}\tag{6.58}$$

where

$$D^{(-)}_0(x-y)=i(2\pi)^{-3}\int d^4k\,\theta(k^0)\delta(k^2)\,e^{ik(x-y)}$$

is the *positive frequency part of the Pauli-Jordan function* of the massless scalar particle.

The *causal Green function* of the graviton is defined as follows:

$$\begin{aligned}D^{c}_{\mu\nu,\rho\sigma}(x-y)&=\langle 0|T[\hat{\varepsilon}_{\mu\nu}(x)\hat{\varepsilon}_{\rho\sigma}(y)]|0\rangle\\&=i^{-1}(2\pi)^{-4}\int d^4p\,e^{-ip(x-y)}\Pi_{\mu\nu,\rho\sigma}(p)(p^2-i\varepsilon)^{-1}\end{aligned}\tag{6.59}$$

or, by means of the differential operator $\hat{\Pi}_{\mu\nu,\rho\sigma}(\Box,\partial^2_{xy})$,

$$D^{c}_{\mu\nu,\rho\sigma}(x-y)=\hat{\Pi}_{\mu\nu,\rho\sigma}(\Box,\partial^2_{xy})D^{c}_0(x-y)\tag{6.60}$$

It is natural that $D^c_0(x-y)$ is the causal Green function of the massless scalar particle given by the standard form

$$D^c_0(x-y)=i^{-1}(2\pi)^{-4}\int d^4k\,e^{-ik(x-y)}(k^2-i\varepsilon)^{-1}\tag{6.61}$$

The *retarded* and *advanced Green functions* can be defined in the following way:

$$\begin{aligned}D^{\text{ret}}_{\mu\nu,\rho\sigma}(x)&=\theta(x^0)D_{\mu\nu,\rho\sigma}(x)=D^{c}_{\mu\nu,\rho\sigma}(x)+D^{(+)}_{\mu\nu,\rho\sigma}(x)\\D^{\text{adv}}_{\mu\nu,\rho\sigma}(x)&=-\theta(-x^0)D_{\mu\nu,\rho\sigma}(x)=D^{c}_{\mu\nu,\rho\sigma}(x)-D^{(-)}_{\mu\nu,\rho\sigma}(x)\end{aligned}\tag{6.62}$$

They satisfy the conditions

$$D^{\text{ret}}_{\mu\nu,\rho\sigma}(x)=0\qquad\text{for}\quad\begin{cases}x^2<0\\x^2>0,\quad x^0<0\end{cases}$$

$$D^{\text{adv}}_{\mu\nu,\rho\sigma}(x)=0\qquad\text{for}\quad\begin{cases}x^2<0\\x^2>0,\quad x^0>0\end{cases}$$

Thus, we can see that all Green functions satisfy all requirements of the local quantum field theory. The following correlations are valid for the Green functions $D^{c}_{\mu\nu,\rho\sigma}(x)$ and $D^{(\pm)}_{\mu\nu,\rho\sigma}(x)$

$$\begin{aligned} D^{c}_{\mu\nu,\rho\sigma}(x) &= \theta(x^0)D^{(-)}_{\mu\nu,\rho\sigma}(x)+\theta(-x^0)D^{(+)}_{\mu\nu,\rho\sigma}(x) \\ D^{c*}_{\mu\nu,\rho\sigma}(x) &= \theta(x^0)D^{(+)}_{\mu\nu,\rho\sigma}(x)+\theta(-x^0)D^{(-)}_{\mu\nu,\rho\sigma}(x) \end{aligned} \tag{6.63}$$

It is easy to observe directly that, similar to the case of the usual theory of quantized fields, in the gravitational theory with the above-defined Green functions (or, equivalently, with the quantum fluctuation of the space-time metric) there exist ultraviolet divergences connected with singularities of these functions, i.e., functions $D^{(\pm)}_{\mu\nu,\rho\sigma}(x)$ and $D^{c}_{\mu\nu,\rho\sigma}(x)$ not defined on the light cone $x^2=0$. To remove the ultraviolet divergences from our theory we use the hypothesis of the existence of a *fundamental length* in nature and change the Green functions in accordance with the nonlocal or stochastic regularization method employed in Namsrai (1986*b*).

6.6. The Change of the Newtonian Law and Form Factors of the Theory

To construct a theory with quantum fluctuations in the space-time metric we introduce a fundamental length into the physical processes by means of a change of the *Newtonian law* at short distances. We assume that in the static limit the Newtonian potential is given by

$$\begin{aligned} \varphi(\mathbf{r}) &= G(2\pi)^{-3}\int d^3p\, e^{-i\mathbf{pr}}[\eta^{\mu\rho}\eta^{\nu\delta}\Pi_{\mu\nu,\rho\delta}(\mathbf{p})]\mathbf{p}^{-2}/10 \\ &= G/4\pi r \end{aligned} \tag{6.64}$$

Such a definition of the potential leads to the idea that if one believes in the changing of the Newtonian law at short distances due to the graviton field carrying information about the space-time structure connected with the existence of the *fundamental length* in nature, then the propagator or the *causal Green function* of the graviton is inescapably modified and in the general case should take the form

$$D^{c}_{\mu\nu,\rho\sigma}(x) = i^{-1}(2\pi)^{-4}\int d^4p\, e^{-ip(x-y)}\Pi_{\mu\nu,\rho\sigma}(p)\,V(p^2l^2)(p^2-i\varepsilon)^{-1} \tag{6.65}$$

where $V(p^2l^2)$ is an arbitrary function of the variable $(-p_0^2+p^2)l^2$, the explicit form of which depends on the concrete method of the regularization procedure. Here we call the parameter l, of the dimension of length, the *fundamental length.* If we employ the *Pauli–Villars regularization* method, then

$$V(p^2l^2)=(1+p^2/\Lambda^2)^{-\nu}, \qquad \nu\geq 2$$

where $\Lambda \equiv M = \hbar / lc$ plays the role of the universal mass value. Thus, a multiplier of the type of

$$p^{-2}(1+p^2l^2)^{-1} \qquad \text{or} \qquad p^{-2}(1+p^2l^2)^{-2}$$

may be obtained in (6.65) due to the following formal procedure:

$$(1) \qquad p^{-2}-(l^{-2}+p^2)^{-1}=p^{-2}(1+p^2l^2)^{-1} \tag{6.66a}$$

$$(2) \qquad p^{-2}-(l^{-2}+p^2)^{-1}-2l^{-2}(l^{-2}+p^2)^{-2}=p^{-2}(1+p^2l^2)^{-2} \tag{6.66b}$$

and so on.

Notice that, on the other hand, in accordance with the scheme formulated in Efimov (1977, 1985) and Namsrai (1986*b*), formula (6.65) is equivalent to introducing a nonlocal graviton field, and the vacuum expectation value of its T-product operators gives just the modified propagator (6.65) of the theory, where $V(p^2l^2)$ is an entire analytic function of the argument $z=p^2l^2$ and decreases rapidly enough in the Euclidean direction of the variable $z\to\infty$.

We distinguish here some possible versions of the Newtonian potential depending on the concrete form of the form factor of the theory. For example, without loss of generality let us consider the Pauli–Villars regularization prescription:

$$\begin{aligned}
&\varphi_1(\mathbf{r})=(G/4\pi r)(1-e^{-r/l}), && \text{for } V_1(p^2l^2)=(1+p^2l^2)^{-1}\\
&\varphi_2(\mathbf{r})=(G/4\pi r)[1-\cos(r/l\sqrt{2})\, e^{-r/l\sqrt{2}}], && \text{for } V_2(p^2l^2)=(1+p^4l^4)^{-1}\\
&\varphi_3(\mathbf{r})=(G/4\pi r)[1-\tfrac{1}{2}(2+r/l)\, e^{-r/l}] && \text{for } V_3(p^2l^2)=(1+p^2l^2)^{-2}\\
&\varphi_4(\mathbf{r})=(G/4\pi r)\{1-\tfrac{1}{8}[8+5r/l+(r/l)^2]\, e^{-r/l}\} && \text{for } V_4(p^2l^2)=(1+p^2l^2)^{-3}
\end{aligned} \tag{6.67}$$

Now we calculate the Green function (6.59) at the point $x^2=0$ for these form factors $V_i(p^2l^2)$. Thus,

$$D^c_{\mu\nu,\rho\sigma}(0)=\tfrac{5}{9}(\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}-\tfrac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma})\tilde{D}_q(0) \tag{6.68}$$

where the function

$$\tilde{D}_q(0)=G(2\pi)^{-4}\int d^4p\, p^{-2}V(p^2l^2)$$

$$=G(2\pi)^{-4}\times\begin{Bmatrix}\infty & \text{for} & V_1(p^2l^2)\\ \pi^3l^{-2} & \text{for} & V_2(p^2l^2)\\ \pi^2l^{-2} & \text{for} & V_3(p^2l^2)\\ \pi^2l^{-2}/2 & \text{for} & V_4(p^2l^2)\end{Bmatrix} \tag{6.69}$$

is finite in the light cone $x^2=0$.

The theory with entire analytic form factors was considered by Efimov (1977, 1985) and Namsrai (1986*b*). Dineykhan and Namsrai (1989) present the stochastic quantization method based on random field-like white noise with nonlocal distributions leading to the appearance of entire form factors. At present, a synonymous definition of the form of the form factors of the theory is not known, and needs a deeper study in this direction as well as another fundamental physical principle (or complete physical information about possible verification of the Newtonian law at small distances).

6.7. Definition of Physical Quantities in the Theory with Quantum Metric

6.7.1. T-Product and Vacuum Expectation of Quantized Quantities

As shown above, introduction of the quantized field $\hat{\varepsilon}_{\mu\nu}(x)$ of (6.1) into our scheme leads to a theory with a quantum fluctuation of the space-time metric (6.33). We observe that in such a theory all physical quantities become operator-valued ones through the metric tensor. Now we address the question of how to define these quantities in space-time with the quantum fluctuating metric. Our basic assumptions are the following:

1. Let F be any physical quantity; then, by definition, its value in space-time with a quantum fluctuating metric acquires the form

$$F \Rightarrow \hat{F}[\varepsilon] = T\hat{F} \tag{6.70}$$

where T denotes the *T-product symbol* acting on the quantized fields $\hat{\varepsilon}_{\mu\nu}(x)$ entering into the quantity $\hat{F}$ through the metric tensor.

2. The averaging procedure for the quantity $\hat{F} = \hat{F}[\varepsilon]$ is reduced to taking its *vacuum expectation value*

$$F_{\text{Ph}} = \langle \hat{F}[\varepsilon] \rangle_q = \langle 0 | T\hat{F} | 0 \rangle \tag{6.71}$$

We call the latter an averaged or observable value of the operator-valued quantity $\hat{F}[\varepsilon]$. Now let us list some properties of the T-product operations of operators $\hat{\varepsilon}_{\mu\nu}(x)$ [for details, see Namsrai (1986*b*, Section 2.5)].

3. Let us introduce the operator $\hat{R}[\varepsilon]$:

$$\hat{R}[\varepsilon] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots \int d^4x_n \, R_n(x_1, \ldots, x_n) T[\hat{\varepsilon}(x_1) \cdots \hat{\varepsilon}(x_n)] \tag{6.72}$$

The operator $\hat{R}[\varepsilon]$ is defined by a set of functions $\{R_n(x_1, \ldots, x_n)\}$. Let us define the *operation of conjugation*

$$[\hat{\varepsilon}_{\mu\nu}(x)]^* = \hat{\varepsilon}_{\mu\nu}(x)$$

Then, for the operator $\hat{R}[\varepsilon]$ in (6.72) we obtain the expression

$$\hat{R}^*[\varepsilon] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots \int d^4x_n \, R_n^*(x_1, \ldots, x_n) T[\hat{\varepsilon}(x_1) \cdots \hat{\varepsilon}(x_n)]$$

4. Now we determine the *operation of multiplication* of two operators $\hat{R}_1[\varepsilon]$ and $\hat{R}_2[\varepsilon]$ of the type of (6.72) by definition:

$$\begin{aligned}\hat{R}_1[\varepsilon] \circledast \hat{R}_2[\varepsilon] &\overset{\text{def}}{=} T\{\hat{R}_1[\varepsilon]\hat{R}_2[\varepsilon]\}\\ &= \sum_{n,m} \frac{1}{n!\,m!} \int d^4x_1 \cdots \int d^4x_n\\ &\quad\times \int d^4y_1 \cdots \int d^4y_m\, R_n(x_1, \ldots, x_n)\\ &\quad\times R_m(y_1, \ldots, y_m) T[\hat{\varepsilon}(x_1) \cdots \hat{\varepsilon}(x_n)\hat{\varepsilon}(y_1) \cdots \hat{\varepsilon}(y_m)]\end{aligned} \tag{6.73}$$

6.7.2. The Operator-Valued Transformation of Coordinates and the Special Theory with Quantized Field

To obtain the quantum fluctuating metric form (6.33) or (6.38), consider the operator-valued transformation (1.3) and its Jacobian (1.10). As above, the condition $d^2\xi^\alpha/d\tau^2 = 0$ leads to the equation

$$\frac{d^2x^\lambda}{d\tau^2} + \hat{\gamma}^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0 \tag{6.74}$$

if we use of the definition

$$\hat{\gamma}^\lambda_{\mu\nu} = T\left\{\frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu\, \partial x^\nu}\right\} \tag{6.75}$$

which ensures the symmetric property of the quantized *affine* connection $\hat{\gamma}^\lambda_{\mu\nu} = \hat{\gamma}^\lambda_{\nu\mu}$. In this case, the *proper time* (quantized) is defined as follows:

$$d\tau^2 = -\hat{g}_{\mu\nu}\, dx^\mu\, dx^\nu \tag{6.76}$$

where

$$\hat{g}_{\mu\nu} = \eta_{\alpha\beta} T\left\{\frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu}\right\} \tag{6.77}$$

The latter yields the metric tensor (6.38).

The multiplication rule (6.73) allows us to establish the connection between quantized quantities $\hat{\gamma}^\sigma_{\mu\nu}$ and $\hat{g}_{\mu\nu}$:

$$\hat{\gamma}^\sigma_{\lambda\mu} = \tfrac{1}{2} T\{\hat{g}^{\nu\sigma}[\partial \hat{g}_{\mu\nu}/\partial x^\lambda + \partial \hat{g}_{\lambda\nu}/\partial x^\mu - \partial \hat{g}_{\mu\lambda}/\partial x^\nu]\} \tag{6.78}$$

where we have used the definitions

$$T\left(\frac{\partial \xi^\beta}{\partial x^\lambda}\right) \circledast T\left\{\frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu\, \partial x^\nu}\right\} \overset{\text{def}}{=} T\left\{\frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu\, \partial x^\nu} \frac{\partial \xi^\beta}{\partial x^\lambda}\right\}$$

and

$$T\left\{\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial \xi^\alpha}{\partial x^\rho}\right\}=T\left\{\frac{\partial \xi^\lambda}{\partial x^\beta}\frac{\partial x^\beta}{\partial \xi^\rho}\right\}=\delta^\lambda_\rho$$

since one can rearrange operators entering into the T-product operation. In the relation (6.77) the inverse metric tensor (quantized) $\hat{g}^{\nu\sigma}$ is defined as

$$\hat{g}^{\nu\sigma}=\eta^{\alpha\beta}T\left\{\frac{\partial x^\nu}{\partial \xi^\alpha}\frac{\partial x^\sigma}{\partial \xi^\beta}\right\} \tag{6.79}$$

In the *weak-field limit* it coincides with the expression (6.42). Now it is not difficult to reiterate all the considerations listed in Section 1 by using the definitions (6.70), (6.71), and (6.73). For example, the expression (1.26) and its averaged value (1.29) for the proper time in the quantum fluctuating space-time metric acquire the form

$$\Delta\hat{\tau}=(-\hat{g}_{\mu\nu}\,dx^\nu\,dx^\mu)^{1/2}$$

and

$$\Delta\tau_q=\langle 0|\Delta\hat{\tau}|0\rangle=(1-\mathbf{v}^2/c^2)^{1/2}[1+\tfrac{5}{24}\tilde{D}_q(0)] \tag{6.80}$$

where $\hat{g}_{\mu\nu}(x)$ and $\tilde{D}_q(0)$ are given by (6.77) [or (6.38)] and (6.69), respectively. In the given case, the square of the *spatial distance* is also defined,

$$d\hat{l}^2=\hat{\gamma}_{ij}\,dx^i\,dx^j$$

with

$$\hat{\gamma}_{ij}=T\{\hat{g}_{ij}-\hat{g}_{0i}\hat{g}_{0j}/\hat{g}_{00}\}$$

and in the weak-field limit its *vacuum expectation value* is

$$dl_q^2=\langle 0|d\hat{l}^2|0\rangle=dl_0^2(1+\tfrac{5}{72}\tilde{D}_q(0)) \tag{6.81}$$

Space-time with a quantum fluctuation in the metric also gives rise to the appearance of an additional potential

$$\hat{\varphi}_f=\tfrac{1}{2}c^2(1-\hat{g}_{00}-\tfrac{1}{2}\hat{\varepsilon}_{00}^2) \tag{6.82}$$

and to the *changing of the particle energy*

$$\hat{\mathscr{E}}=mc^2(1-\mathbf{v}^2/c^2)^{-1/2}(-\hat{g}_{00})^{1/2} \tag{6.83}$$

and their vacuum expectation values are given by the formulas (1.73) and (1.75), where $\tilde{D}(0)$ should be replaced by $\tilde{D}_q(0)$ defined in (6.69).

6.8. Modified Gravitational Theory with Quantum Fluctuating Metric

It is not difficult to reconstruct the general theory of relativity with a quantum fluctuation of the space-time metric. For this purpose, we use results obtained in Sections 2-5, where the space-time metric has been regarded as a stochastic quantity. In the given case, the general equivalence principle is easily reformulated in accordance with the method expounded in Section 2. For example, using the definitions (6.70) and (6.71), we can obtain the analogous equation to (2.6), that is,

$$\frac{d^2 z^\lambda}{d\tau^2} + \hat{\Gamma}^\lambda_{\mu\nu} \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} = 0 \tag{6.84}$$

where the *quantized affine connection* is given by

$$\hat{\Gamma}^\sigma_{\lambda\mu} = \tfrac{1}{2} T\{\hat{G}^{\nu\sigma}[\partial \hat{G}_{\mu\nu}/\partial z^\lambda + \partial \hat{G}_{\lambda\nu}/\partial z^\mu - \partial \hat{G}_{\mu\lambda}/\partial z^\nu]\} \tag{6.85}$$

and, by definition, the metric tensor $\hat{G}_{\mu\nu}$ reads

$$\hat{G}_{\mu\nu}(z) = T\left\{\frac{\partial \xi^\rho}{\partial z^\mu} \frac{\partial \xi^\delta}{\partial z^\nu}\right\} \eta_{\rho\delta} = T\left\{\hat{g}_{\alpha\beta} \frac{\partial x^\alpha}{\partial z^\mu} \frac{\partial x^\beta}{\partial z^\nu}\right\}$$

$$= g^0_{\mu\nu}(z) + \hat{\varepsilon}_{\mu\nu}(z) + \tfrac{1}{4} T\{\hat{\varepsilon}^\rho_\mu(z) \hat{\varepsilon}_{\nu\rho}(z)\} \tag{6.86}$$

Here

$$g^0_{\mu\nu}(z) = \eta_{\alpha\beta} \frac{\partial x^\alpha}{\partial z^\mu} \frac{\partial x^\beta}{\partial z^\nu}$$

is responsible for a purely external gravitational field and becomes $\eta_{\mu\nu}$ when the latter is absent ($z^\nu \equiv x^\nu$).

In the Newtonian approximation, equation (6.84) has the standard form

$$\frac{d^2 \mathbf{z}}{dt^2} = \tfrac{1}{2} \nabla H_{00} \tag{6.87}$$

where

$$H_{00} = -2\phi_N + \hat{\varepsilon}_{00}(z) + \tfrac{1}{4} T\{\hat{\varepsilon}^\rho_0(z) \hat{\varepsilon}_{0\rho}(z) + 2\hat{\varepsilon}^2_{00}(z)\}$$

and the space-time metric is

$$\hat{G}_{00} = -1 - 2\phi_N + \hat{\varepsilon}_{00}(z) + \tfrac{1}{4} T\{\hat{\varepsilon}^\rho_0(z) \hat{\varepsilon}_{0\rho}(z) + 2\hat{\varepsilon}^2_{00}(z)\}$$

In the latter case, the potential force is given by the vacuum expectation value:

$$\mathbf{F}_q = \langle 0|\hat{\mathbf{F}}|0\rangle = \langle 0|T\{1 + \tfrac{3}{4}\hat{\varepsilon}_{00}(z)\hat{\varepsilon}_{00}(z) + \tfrac{1}{4}\hat{\varepsilon}^\rho_0(z)\hat{\varepsilon}_{0\rho}(z) - 2\phi_N\}|0\rangle \cdot \mathbf{F}_N$$

or

$$\mathbf{F}_q = [1 + \tfrac{3}{4} D^c_{00,00}(0) + \tfrac{1}{4} D^{c\rho;}_{0,0\rho}(0) - 2\phi_N]\mathbf{F}_N \tag{6.88}$$

where $\mathbf{F}_N = -\nabla\phi_N$ is the *Newtonian force* and $D^c_{00,00}(0)$ is the value of the propagator of the graviton at the point $x = 0$.

Notice that due to the quantum fluctuation of the space-time metric, a contribution to the *red-shift* value also occurs and its vacuum expectation is

$$\begin{aligned}\langle \Delta\nu/\nu\rangle_q = {}& \phi_N(z_2)-\phi_N(z_1)+\tfrac{1}{2}[\phi_N(z_2)-\phi_N(z_1)]^2\\ &-[\phi_N^2(z_2)-\phi_N^2(z_1)]-\tfrac{1}{4}D^c_{00,00}(0)[-1-3\phi_N(z_2)+7\phi_N(z_1)]\\ &-\tfrac{1}{4}D^{c\rho}_{0,0\rho}(0)[\phi_N(z_1)-\phi_N(z_2)]\\ &-\tfrac{1}{4}D^c_{00,00}(z_1-z_2)[1-\phi_N(z_2)-3\phi_N(z_1)]\end{aligned}$$

Even in the absence of the external gravitational field defined by the Newtonian potential $\phi_N(z)$, the red-shift contribution due to the pure quantized radiation field $\hat{\varepsilon}_{\mu\nu}(z)$ remains:

$$\langle \Delta\nu/\nu\rangle_q = \tfrac{1}{4}D^c_{00,00}(0)-\tfrac{1}{4}D^c_{00,00}(z_1-z_2)$$

which is the standard form in our scheme.

6.9. Tensor Analysis in Space-Time with Quantum Metric

By using the T-product definition of physical quantities, the tensor analysis in space-time with quantized metric $\hat{G}_{\mu\nu}(z)$ and operator-valued affine connection $\Gamma^{\lambda}_{\mu\nu}(z)$ are easily reconstructed in accordance with Section 3. As above, in the given case, we have at our disposal three systems of reference:

(a) The local inertial system of reference ξ^α with the Minkowski metric $\eta_{\alpha\beta}$.

(b) The "quasilocal" or "quasi-quantum inertial" system of reference x^ν with the quantized metric

$$\hat{g}_{\mu\nu}(x)=\eta_{\mu\nu}+\hat{\varepsilon}_{\mu\nu}+\tfrac{1}{4}T[\hat{\varepsilon}^{\rho}_{\nu}(x)\hat{\varepsilon}_{\mu\rho}(x)]$$

(c) The general system of reference z^μ with the metric (quantum) $\hat{G}_{\mu\nu}$.

Thus, the chain rule is valid for the force $\hat{F}^\mu_q$:

$$\hat{F}^\mu_q(z)=T\left\{\frac{\partial z^\mu}{\partial x^\nu}F^\nu(x)\right\}=T\left\{\frac{\partial z^\mu}{\partial x^\nu}\frac{\partial x^\nu}{\partial \xi^\alpha}f^\alpha_f\right\}$$

Since the operator-valued transformation matrix $\partial x^\nu/\partial\xi^\alpha$ with the quantized field $\hat{\varepsilon}_{\mu\nu}(x)$ is given by the analogous formula (1.12), one gets

$$\begin{aligned}\hat{F}^\mu_q(z)&=T\left\{\frac{\partial z^\mu}{\partial x^\nu}[\delta^\nu_\alpha-\tfrac{1}{2}\hat{\varepsilon}^\nu_\alpha(x)+\tfrac{1}{4}\hat{\varepsilon}^\rho_\alpha(x)\hat{\varepsilon}^\nu_\rho(x)-\cdots]f^\alpha_f\right\}\\ &=\frac{\partial z^\mu}{\partial x^\nu}f^\nu_f+\frac{\partial z^\mu}{\partial x^\nu}f^\alpha_f T\{-\tfrac{1}{2}\hat{\varepsilon}^\nu_\alpha+\tfrac{1}{4}\hat{\varepsilon}^\rho_\alpha(x)\varepsilon^\nu_\rho(x)-\cdots\}\end{aligned}$$

in the weak quantum field limit.

Tensor algebra operations in our scheme are the same as in the previous case (Section 3) with the exception of the lowering and raising operations and of the covariant derivatives to be carried out by means of the quantized metric tensor $\hat{G}_{\mu\nu}$ and the affine connection $\hat{\Gamma}^{\lambda}_{\mu\nu}$ respectively.

It is natural to define the determinant of the quantized tensor $G_{\mu\nu}$ by the formula

$$G_q = -\langle 0|\text{Det } \hat{G}_{\mu\nu}|0\rangle \tag{6.89}$$

since in the definition of $\hat{G}_{\mu\nu}$, the T-product symbol is involved.

6.10. Gravitational Effects and the Einstein Equation in Space-Time with Quantum Metric

Mechanical and electromagnetic processes in the presence of gravity with the quantized metric $\hat{G}_{\mu\nu}$ may be considered by the same method as in Section 4, where we should change the stochastic metric by the quantum one $\hat{G}_{\mu\nu}$, the affine connection $\Gamma^{\lambda}_{\mu\nu}$ by its quantized version $\hat{\Gamma}^{\lambda}_{\mu\nu}$, and the determinant of $G_{\mu\nu}$ by its vacuum expectation form (6.89). Here the stochastic averaging procedure is replaced by taking the vacuum expectation value for quantized expressions which involve T-product operations.

To construct *Einstein's equation* in space-time with the quantized metric $\hat{G}_{\mu\nu}$ we use the definition of the curvature tensor (5.5), whose form in the given case is

$$\hat{R}^{\lambda}_{\mu\nu\varkappa}(z) = \partial\hat{\Gamma}^{\lambda}_{\mu\nu}/\partial z^{\varkappa} - \partial\hat{\Gamma}^{\lambda}_{\mu\varkappa}/\partial z^{\nu} + T[\hat{\Gamma}^{\eta}_{\mu\nu}\hat{\Gamma}^{\lambda}_{\varkappa\eta} - \hat{\Gamma}^{\eta}_{\mu\varkappa}\hat{\Gamma}^{\lambda}_{\nu\eta}] \tag{6.90}$$

By using this *quantized curvature tensor*, the *Ricci tensor* can be defined as

$$\hat{R}_{\mu\varkappa} \equiv \hat{R}^{\lambda}_{\mu\lambda\varkappa} \tag{6.91}$$

and the *scalar curvature*

$$\hat{R} = T[\hat{G}^{\mu\varkappa}\hat{R}_{\mu\varkappa}] \tag{6.92}$$

For further consideration it is convenient to use the following representation:

$$\hat{R}_{\lambda\mu\nu\varkappa} = \tfrac{1}{2}\hat{N}_{\lambda\mu\nu\varkappa} + T\{\hat{G}_{\eta\sigma}[\hat{\Gamma}^{\eta}_{\nu\lambda}\hat{\Gamma}^{\sigma}_{\mu\varkappa} - \hat{\Gamma}^{\eta}_{\varkappa\lambda}\hat{\Gamma}^{\sigma}_{\mu\nu}]\} \tag{6.93}$$

where the quantity $\hat{N}_{\lambda\mu\nu\varkappa}$ is given by (5.12) with the metric $\hat{G}_{\mu\nu}$. In order to define the vacuum expectation values of $\hat{R}_{\mu\nu\varkappa\lambda}$, $\hat{R}_{\mu\varkappa}$, and $\hat{R}$ (or $\hat{G}_{\mu\nu}\hat{R}$) we first give this averaging procedure for the affine connection $\hat{\Gamma}^{\lambda}_{\mu\nu}(z)$, for which an expression of the type of (5.6) is valid:

$$\hat{\Gamma}^{\lambda}_{\mu\nu}(z) = \Gamma^{0\lambda}_{\mu\nu} + T\{-\tfrac{1}{2}\hat{\varepsilon}^{\rho\lambda}(z)\gamma_{\rho;\mu\nu} + \tfrac{3}{8}\hat{\varepsilon}^{\rho\delta}(z)\hat{\varepsilon}^{\lambda}_{\delta}(z)\gamma_{\rho;\mu\nu} + \tfrac{1}{2}g_0^{\rho\lambda}\hat{\mathscr{E}}_{\rho;\mu\nu}$$
$$-\tfrac{1}{2}\hat{\varepsilon}^{\rho\lambda}(z)\hat{\mathscr{E}}_{\rho;\mu\nu} + \tfrac{1}{8}g_0^{\rho\lambda}\hat{E}_{\rho;\mu\nu} + O(\varepsilon^3)\} \tag{6.94}$$

in the *weak quantum field limit*, where the quantities $\gamma_{\rho;\mu\nu}(z)$, $\hat{\mathscr{E}}_{\rho;\mu\nu}(z)$, and $\hat{E}_{\rho;\mu\nu}(z)$ are defined by (5.7) with $\hat{\varepsilon}_{\mu\nu}$. Before averaging an expression of the type of (6.94), it is important to notice that the well-known relation

$$\begin{aligned}\langle 0|T\{\partial_\varkappa\hat{\varepsilon}_{\mu\nu}(x)\,\partial_\lambda\hat{\varepsilon}_{\rho\delta}(y)\}|0\rangle \\ = (\partial^2/\partial x^\varkappa\,\partial y^\lambda)\langle 0|T[\hat{\varepsilon}_{\mu\nu}(x)\hat{\varepsilon}_{\rho\delta}(y)]|0\rangle\end{aligned} \tag{6.95}$$

holds for the T-product of quantized fields $\hat{\varepsilon}_{\mu\nu}(x)$. In other words, the T-product in the Wick sense coincides with the T-product in the Dyson sense, i.e., symbolically $T_{\mathrm{W}} = T_{\mathrm{D}}$. Thus, the following correlations take place:

$$\begin{aligned}\partial_\varkappa\partial_\lambda D^c_{\mu\nu,\rho\sigma}(x) &= \theta(x^0)\partial_\varkappa\partial_\lambda D^{(-)}_{\mu\nu,\rho\sigma}(x) + \theta(-x^0)\,\partial_\varkappa\partial_\lambda D^{(+)}_{\mu\nu,\rho\sigma}(x) \\ \partial_\varkappa\partial_\lambda D^{(-)}_{\mu\nu,\rho\sigma}(x) &= \theta(x^0)\partial_\varkappa\partial_\lambda D^{c}_{\mu\nu,\rho\sigma}(x) + \theta(-x^0)\partial_\varkappa\partial_\lambda D^{c^*}_{\mu\nu,\rho\sigma}(x)\end{aligned} \tag{6.96}$$

From the relation (6.95) and the explicit form of the causal Green function (6.65) it follows that

$$\begin{aligned}\langle 0|T[\partial_\varkappa\hat{\varepsilon}_{\mu\nu}(z)\cdot\hat{\varepsilon}_{\rho\sigma}(z)]|0\rangle \\ = -(2\pi)^{-4}\int d^4p\,\Pi_{\mu\nu,\rho\sigma}(p)\,V(p^2l^2)p_\varkappa(p^2 - i\varepsilon)^{-1}\end{aligned} \tag{6.97}$$

Thus, the vacuum expectation value of (6.94) takes an analogous form to (5.8):

$$\langle 0|\hat{\Gamma}^\lambda_{\mu\nu}(z)|0\rangle = \Gamma^{0\lambda}_{\mu\nu}(z) + \tfrac{3}{8}D^{c\rho\delta,\lambda}_{\delta}(0)\,\gamma_{\rho;\mu\nu} \tag{6.98}$$

where

$$D^{c\rho\delta,\lambda}_{\delta}(0) = \tfrac{5}{2}\eta^{\rho\lambda}\tilde{D}_q(0)$$

and $\tilde{D}_q(0)$ depends on the concrete form of the form factor $V(p^2l^2)$ of the theory, which, in this particular case, is given by (6.69).

Now let us define the *vacuum expectation values* for the *Ricci tensor* $\hat{R}_{\mu\varkappa}$ and the contraction $\hat{R}\hat{G}_{\mu\varkappa}$. The former in the weak-field approximation acquires an analogous form to (5.18), where the stochastic field $\varepsilon_{\mu\nu}(z)$ should be replaced by the quantized one $\hat{\varepsilon}_{\mu\nu}(z)$ and the T-product operation must also be involved. After taking the averaging procedure in accordance with (6.59) and (6.95), the contribution to $R^0_{\mu\varkappa}$ due to the quantized metric $\hat{G}_{\mu\nu}$ depends on the functions $D^c_{\mu\nu,\rho\delta}(0)$ and $D^c_{1\rho\sigma,\mu\varkappa;\nu\lambda}(0)$ defined by the same formula (5.25), where one needs to replace $\delta_{\mu\rho} \Rightarrow \eta_{\mu\rho}$, $\tilde{D}(0) \Rightarrow \tilde{D}_q(0)$, and

$$\tilde{D}_1(0) \Rightarrow \tilde{D}_{1q}(0) = G(2\pi)^{-4}\int d^4p_E\;V(p_E^2l^2) < \infty \tag{6.99}$$

Thus, after elementary calculations as in Section 5.3, we have

$$\langle 0|\hat{R}_{\mu\varkappa}|0\rangle = R^0_{\mu\varkappa} + \langle 0|\hat{Q}_{\mu\varkappa}|0\rangle \tag{6.100}$$

where

$$\begin{aligned}\langle 0|\hat{Q}_{\mu\varkappa}|0\rangle = {} & \tfrac{15}{8}\tilde{D}_q(0)N^1_{\underline{\lambda}\mu\underline{\lambda}\varkappa} + \frac{1}{2}\left(\frac{64}{a}+\frac{13}{6}\right)\tilde{D}_{1q}(0)\eta_{\mu\varkappa} + \tfrac{5}{18}\tilde{D}_q(0) \\ & \times[2\gamma_{\underline{\lambda};\underline{\nu}\underline{\lambda}}\gamma_{\underline{\nu};\mu\varkappa} - \gamma_{\underline{\rho};\varkappa\underline{\rho}}\gamma_{\underline{\nu};\mu\underline{\nu}} - \gamma_{\underline{\rho};\mu\underline{\nu}}\gamma_{\underline{\nu};\varkappa\underline{\rho}} - \tfrac{1}{2}\gamma_{\underline{\rho};\underline{\lambda}\underline{\lambda}}\gamma_{\underline{\rho};\mu\varkappa} \\ & + \tfrac{1}{2}\gamma_{\underline{\rho};\varkappa\underline{\lambda}}\gamma_{\underline{\rho};\mu\underline{\lambda}} - 9g^{0\nu\lambda}(\gamma_{\rho;\nu\lambda}\Gamma^{0\rho}_{\mu\varkappa} - \gamma_{\rho;\varkappa\lambda}\Gamma^{0\rho}_{\mu\nu})] \\ & + \tfrac{5}{9}\tilde{D}_q(0)\{\tfrac{27}{8}g^0_{\eta\sigma}(\Gamma^{0\eta}_{\underline{\lambda}\underline{\lambda}}\Gamma^{0\sigma}_{\mu\varkappa} - \Gamma^{0\eta}_{\varkappa\underline{\lambda}}\Gamma^{0\sigma}_{\mu\underline{\lambda}}) \\ & - [2\Gamma^{0\lambda}_{\nu\lambda}\Gamma^{0\nu}_{\mu\varkappa} - \Gamma^{0\lambda}_{\varkappa\lambda}\Gamma^{0\nu}_{\mu\nu} - \Gamma^{0\nu}_{\varkappa\lambda}\Gamma^{0\lambda}_{\mu\nu} - \tfrac{1}{2}(\Gamma^{0\eta}_{\lambda\lambda}\Gamma^{0\eta}_{\mu\varkappa} - \Gamma^{0\eta}_{\varkappa\lambda}\Gamma^{0\eta}_{\mu\lambda})] \\ & + \tfrac{9}{8}g^{0\lambda\nu}(\Gamma^{0\underline{\eta}}_{\nu\lambda}\Gamma^{0\underline{\eta}}_{\mu\varkappa} - \Gamma^{0\underline{\eta}}_{\varkappa\lambda}\Gamma^{0\underline{\eta}}_{\mu\nu})\} + \tfrac{5}{9}\tilde{D}_q(0)g^{0\nu\lambda} \\ & \times\{\tfrac{27}{16}(\gamma_{\underline{\eta};\nu\lambda}\gamma_{\underline{\eta};\mu\varkappa} - \gamma_{\underline{\eta};\varkappa\lambda}\gamma_{\underline{\eta};\mu\nu}) + \tfrac{1}{4}[g^0_{\underline{\eta}\underline{\eta}}(\gamma_{\underline{\rho};\nu\lambda}\gamma_{\underline{\rho};\mu\varkappa} - \gamma_{\underline{\rho};\varkappa\lambda}\gamma_{\underline{\rho};\mu\nu}) \\ & + \tfrac{1}{2}g^0_{\underline{\eta}\underline{\sigma}}(\gamma_{\underline{\sigma},\nu\lambda}\gamma_{\underline{\eta};\mu\varkappa} - \gamma_{\underline{\sigma};\varkappa\lambda}\gamma_{\underline{\eta};\mu\nu})]\} + \tfrac{5}{288}\tilde{D}_{1q}(0) \\ & \times[31\bar{g}^{0\rho\rho}\bar{g}^0_{\mu\varkappa} - 20\bar{g}^0_{\mu\varkappa} - 14(\bar{g}^{0\rho\rho})^2\eta_{\mu\varkappa} + 10\bar{g}^0_{\rho\rho'}g^{0\rho\rho'}\eta_{\mu\varkappa}]\end{aligned} \tag{6.101}$$

where we have used the notation

$$\bar{g}^0_{\mu\varkappa} \equiv g^{0\nu\lambda}\eta_{\nu\varkappa}\eta_{\lambda\mu}, \qquad \bar{g}^{0\rho\rho} \equiv g^{0\rho\rho'}\eta_{\rho\rho'}$$

and have distinguished two types of summation over indices:

$$\gamma_{\underline{\eta};\nu\lambda}\gamma_{\underline{\eta};\mu\varkappa} \equiv \eta_{\eta\rho}\gamma_{\eta;\nu\lambda}\gamma_{\rho;\mu\varkappa} \qquad \text{or} \qquad \gamma_{\underline{\lambda};\underline{\nu}\underline{\lambda}}\gamma_{\underline{\nu};\mu\varkappa} \equiv \eta_{\lambda\delta}\eta_{\nu\alpha}\gamma_{\lambda;\nu\delta}\gamma_{\alpha;\mu\varkappa}$$

and

$$\Gamma^{0\lambda}_{\nu\lambda}\Gamma^{0\nu}_{\mu\varkappa} = \textstyle\sum_{\lambda,\nu=0,1,2,3}\Gamma^{0\lambda}_{\nu\lambda}\Gamma^{0\nu}_{\mu\varkappa} = \delta^{\lambda\rho}\delta^{\alpha\nu}\Gamma^{0\lambda}_{\nu\rho}\Gamma^{0\alpha}_{\mu\varkappa}$$

since the Euclidean metric $\delta^{\lambda\rho} = \delta^\rho_\lambda$ possesses the property

$$\delta^\nu_\lambda T^\lambda = T^\nu \qquad \text{or} \qquad \delta^\nu_\lambda T^{\lambda\rho}_\sigma = T^{\nu\rho}_\sigma$$

for any tensor quantities T^ν and $T^{\nu\rho}_\sigma$.

An analogous calculation yields

$$\begin{aligned}\langle 0|T[\hat{G}_{\mu\varkappa}\hat{R}]|0\rangle = {} & R^0 g^0_{\mu\varkappa} + \tfrac{5}{8}R^0\tilde{D}_q(0)\eta_{\mu\varkappa} - \tfrac{5}{9}\tilde{D}_q(0)(2R^0_{\mu\varkappa} - \tfrac{1}{2}\eta_{\mu\varkappa}\bar{R}^0) \\ & + \tfrac{15}{8}\tilde{D}_q(0)g^0_{\mu\varkappa}\bar{R}^0 + g^{0\beta\beta'}\langle 0|T[\hat{\varepsilon}_{\mu\varkappa}\hat{M}_{\beta\beta'}]|0\rangle \\ & - g^0_{\mu\varkappa}\{g^{0\beta\beta'}\langle 0|\hat{Q}_{\beta\beta'}|0\rangle - \langle 0|T[\hat{\varepsilon}^{\beta\beta'}\hat{M}_{\beta\beta'}]|0\rangle\}\end{aligned} \tag{6.102}$$

where $\bar{R}^0 = \eta^{\beta\beta'}R^0_{\beta\beta'}$, and $\langle 0|\hat{Q}_{\beta\beta'}|0\rangle$ is given by formula (6.101). Therefore, it only remains to calculate the term of the type of $\langle 0|T[\hat{\varepsilon}_{\mu\varkappa}\hat{M}_{\beta\beta'}]|0\rangle$, where the expression $\hat{M}_{\beta\beta'}$ is defined by (5.19a) with the quantized field $\hat{\varepsilon}_{\mu\varkappa}$. Now we calculate the term

$$H_{\mu\varkappa} = g^{0\beta\beta'}\langle 0|T[\hat{\varepsilon}_{\mu\varkappa}\hat{M}_{\beta\beta'}]|0\rangle$$

Taking into account the explicit form (5.19a) for the quantized expression $\hat{M}_{\beta\beta'}$, we have

$$\begin{aligned}H_{\mu\varkappa} = &-\tfrac{1}{2}N^1_{\lambda\beta\nu\beta'}g^{0\beta\beta'}\langle 0|T[\hat{\varepsilon}^{\nu\lambda}\hat{\varepsilon}_{\mu\varkappa}]|0\rangle + \tfrac{1}{2}g^{0\nu\lambda}g^{0\beta\beta'}\langle 0|T[\hat{\varepsilon}_{\mu\varkappa}N^2_{\beta\lambda\nu\beta'}]|0\rangle\\ &+g^{0\nu\lambda}g^{0\beta\beta'}g^0_{\eta\sigma}\langle 0|T[\hat{\varepsilon}_{\mu\varkappa}\hat{\Lambda}^{\eta\sigma}_{1\beta\lambda\nu\beta'}]|0\rangle\\ &+g^{0\beta\beta'}\langle 0|\{T[\hat{\varepsilon}_{\mu\varkappa}\hat{\varepsilon}_{\eta\sigma}]\cdot g^{0\lambda\nu} - g^0_{\eta\sigma}\cdot T[\hat{\varepsilon}_{\mu\varkappa}\hat{\varepsilon}^{\lambda\nu}]\}|0\rangle\\ &\times(\Gamma^{0\eta}_{\nu\lambda}\Gamma^{0\sigma}_{\beta\beta'} - \Gamma^{0\eta}_{\beta'\lambda}\Gamma^{0\sigma}_{\beta\nu}) \qquad (6.103)\end{aligned}$$

where $\hat{\Lambda}^{\eta\sigma}_{1\beta\lambda\nu\beta'}$ is defined by (5.21). We calculate each term in (6.103) separately. The first term gives

$$\begin{aligned}H^1_{\mu\varkappa} &= -\tfrac{1}{2}g^{0\beta\beta'}N^1_{\lambda\beta\nu\beta'}\cdot\tfrac{5}{9}\tilde{D}_q(0)(\delta^\lambda_\mu\delta^\nu_\varkappa + \delta^\lambda_\varkappa\delta^\nu_\mu - \tfrac{1}{2}\eta^{\lambda\nu}\eta_{\mu\varkappa})\\ &= -\tfrac{5}{18}\tilde{D}_q(0)g^{0\beta\beta'}(N^1_{\mu\beta\varkappa\beta'} + N^1_{\varkappa\beta\mu\beta'} - \tfrac{1}{2}\eta_{\mu\varkappa}N^1_{\lambda\beta\lambda\beta'}),\\ &(N^1_{\lambda\beta\lambda\beta'} = \eta^{\lambda\nu}N^1_{\lambda\beta\nu\beta'})\end{aligned}$$

To obtain the explicit form of the second term $H^2_{\mu\varkappa}$ it is necessary to carry out a long algebraic calculation, with the following result:

$$\begin{aligned}H^2_{\mu\varkappa} = &\tfrac{5}{36}g^{0\lambda\nu}g^{0\beta\beta'}(\eta_{\beta\beta'}\eta_{\mu\lambda}\eta_{\varkappa\nu} + \eta_{\beta\beta'}\eta_{\mu\nu}\eta_{\lambda\varkappa} - \eta_{\beta\beta'}\eta_{\mu\varkappa}\eta_{\lambda\nu}\\ &-\eta_{\beta'\mu}\eta_{\beta\lambda}\eta_{\varkappa\nu} - \eta_{\beta'\mu}\eta_{\beta\nu}\eta_{\lambda\varkappa} - \eta_{\beta'\lambda}\eta_{\beta\varkappa}\eta_{\mu\nu} - \eta_{\beta'\varkappa}\eta_{\beta\nu}\eta_{\mu\lambda}\\ &+2\eta_{\beta'\mu}\eta_{\beta\varkappa}\eta_{\lambda\nu} + \eta_{\beta'\nu}\eta_{\beta\lambda}\eta_{\mu\varkappa})\times\tilde{D}_{1q}(0)\\ = &\tfrac{5}{36}\tilde{D}_{1q}(0)[4\bar{g}^0\bar{g}^0_{\mu\varkappa} - \eta_{\mu\varkappa}(\bar{g}^0)^2 - 4\bar{g}^{0'}_{\mu\varkappa} + \bar{g}^0_{\rho\rho'}g^{0\rho\rho'}\eta_{\mu\varkappa}]\end{aligned}$$

where

$$\bar{g}^0_{\rho\rho'} = g^{0\lambda\nu}\eta_{\nu\rho}\eta_{\lambda\rho'}, \qquad \bar{g}^0 = g^{0\lambda\nu}\eta_{\lambda\nu}, \qquad \bar{g}^{0'}_{\mu\varkappa} = g^0_{\beta\varkappa}g^{0\beta}_{\mu}$$

Taking the vacuum expectation value for other terms in (6.103) is not difficult. They are

$$\begin{aligned}H^3_{\mu\varkappa} = &\tfrac{5}{18}\tilde{D}_q(0)g^{0\beta\beta'}g^{0\lambda\nu}g^0_{\eta\sigma}[(\delta^\rho_\mu\delta^\eta_\varkappa + \delta^\rho_\varkappa\delta^\eta_\mu - \tfrac{1}{2}\eta^{\rho\eta}\eta_{\mu\varkappa})\\ &\times(\gamma_{\rho;\beta'\lambda}\Gamma^{0\sigma}_{\beta\nu} - \gamma_{\rho;\nu\lambda}\Gamma^{0\sigma}_{\beta\beta'}) + (\delta^\rho_\mu\delta^\sigma_\varkappa + \delta^\rho_\varkappa\delta^\sigma_\mu - \tfrac{1}{2}\eta^{\rho\sigma}\eta_{\mu\varkappa})\\ &\times(\gamma_{\rho;\beta\nu}\Gamma^{0\eta}_{\beta'\lambda} - \gamma_{\rho;\beta\beta'}\Gamma^{0\eta}_{\nu\lambda})]\end{aligned}$$

Further, we assume that $g^0_{\eta\sigma}g^{0\rho'\sigma} = \delta^{\rho'}_\eta$, and therefore,

$$g^0_{\eta\sigma}\Gamma^{0\sigma}_{\beta\beta'} = \tfrac{1}{2}\gamma_{\eta;\beta\beta'}$$

with this expression, the term $H^3_{\mu\varkappa}$ takes the form

$$\begin{aligned}H^3_{\mu\varkappa} = &\tfrac{5}{18}\tilde{D}_q(0)g^{0\beta\beta'}g^{0\lambda\nu}[\gamma_{\mu;\beta'\lambda}\gamma_{\varkappa;\beta\nu} + \gamma_{\varkappa;\beta'\lambda}\gamma_{\mu;\beta\nu}\\ &-\gamma_{\mu;\nu\lambda}\gamma_{\varkappa;\beta\beta'} - \gamma_{\varkappa;\nu\lambda}\gamma_{\mu;\beta\beta'}\\ &-\tfrac{1}{2}\eta_{\mu\varkappa}(\gamma_{\eta;\beta\nu}\gamma_{\eta;\beta'\lambda} - \gamma_{\eta;\beta\beta'}\gamma_{\eta;\nu\lambda})]\end{aligned}$$

By the same method as employed above, the last term in (6.103) is easily calculated:

$$\begin{aligned}H^4_{\mu\varkappa} &= \tfrac{5}{9}\tilde{D}_q(0)g^{0\beta\beta'}\{[g^{0\lambda\nu}(\eta_{\eta\mu}\eta_{\sigma\varkappa}+\eta_{\eta\varkappa}\eta_{\mu\sigma}-\tfrac{1}{2}\eta_{\mu\varkappa}\eta_{\eta\sigma})\\ &\quad -g^0_{\sigma\eta}(\delta^\lambda_\mu\delta^\nu_\varkappa+\delta^\lambda_\varkappa\delta^\nu_\mu-\tfrac{1}{2}\eta^{\lambda\nu}\eta_{\mu\varkappa})](\Gamma^{0\eta}_{\nu\lambda}\Gamma^{0\sigma}_{\beta\beta'}-\Gamma^{0\eta}_{\beta'\lambda}\Gamma^{0\sigma}_{\beta\nu})\}\\ &= \tfrac{5}{9}\tilde{D}_q(0)g^{0\beta\beta'}\{\tfrac{1}{4}\eta_{\mu\varkappa}(\Gamma^{0\eta}_{\varrho\varrho}\gamma_{\eta;\beta\beta'}-\Gamma^{0\eta}_{\beta'\varrho}\gamma_{\eta;\beta\varrho})\\ &\quad -\Gamma^{0\eta}_{\mu\varkappa}\gamma_{\eta;\beta\beta'}+\tfrac{1}{2}(\Gamma^{0\eta}_{\beta'\mu}\gamma_{\eta;\beta\varkappa}+\Gamma^{0\eta}_{\beta'\varkappa}\gamma_{\eta;\beta\mu})\\ &\quad +g^{0\lambda\nu}[\Gamma^0_{\mu;\nu\lambda}\Gamma^0_{\varkappa;\beta\beta'}-\Gamma^0_{\mu;\beta'\lambda}\Gamma^0_{\varkappa;\beta\nu}+\Gamma^0_{\varkappa;\nu\lambda}\Gamma^0_{\mu;\beta\beta'}-\Gamma^0_{\varkappa;\beta'\lambda}\Gamma^0_{\mu;\beta\nu}\\ &\quad -\tfrac{1}{2}\eta_{\mu\varkappa}(\Gamma^{0\eta}_{\nu\lambda}\Gamma^{0\eta}_{\beta\beta'}-\Gamma^{0\eta}_{\beta'\lambda}\Gamma^{0\eta}_{\beta\nu})]\}\end{aligned}$$

where

$$\Gamma^{0\delta}_{\nu\lambda}\Gamma^{0\delta}_{\beta\beta'}=\eta_{\rho\varkappa}\Gamma^{0\varkappa}_{\nu\lambda}\Gamma^{0\rho}_{\beta\beta'};\qquad \Gamma^0_{\mu;\nu\lambda}=\eta_{\mu\sigma}\Gamma^{0\sigma}_{\nu\lambda}$$

An analogous calculation may be made for the last term in (6.102). Thus, all the necessary terms in (6.102) are calculated by using (6.101) and (6.103).

Now we are able to rewrite *Einstein's equation* in space-time with the quantized metric $\hat{G}_{\mu\nu}$. The expected formula for this equation is

$$T\{\hat{R}_{\mu\nu}-\tfrac{1}{2}\hat{G}_{\mu\nu}\hat{R}\}=-8\pi G\hat{T}_{\mu\nu} \tag{6.104}$$

In accordance with the general covariance principle formulated in Sections 3-5, the energy-momentum tensor $\hat{T}_{\mu\nu}$ in our scheme is defined as

$$\hat{T}_{\mu\nu}(z)=T\left\{\frac{\partial x^\alpha}{\partial z^\mu}\frac{\partial x^\beta}{\partial z^\nu}T_{\alpha\beta}\right\}=\frac{\partial x^\alpha}{\partial z^\mu}\frac{\partial x^\beta}{\partial z^\nu}T^f_{\eta\delta}T\left(\frac{\partial\xi^\eta}{\partial x^\alpha}\frac{\partial\xi^\delta}{\partial x^\beta}\right)$$

Here the operator-valued Jacobian of transformation $\partial\xi^\eta/\partial x^\alpha$ is given by (1.10) with the quantized field $\hat{\varepsilon}^\alpha_\mu(x)$.

The vacuum expectation of equation (6.104) may be easily calculated by using expressions (6.100) and (6.102). In the case of the quantum fluctuating space-time metric the Bianchi identities (5.42)-(5.44) and the coordinate condition (5.45) are also fulfilled if we choose the T-product operation in the definition of the operator-valued quantities $\hat{G}^{\mu\nu}$, $\hat{\Gamma}^\lambda_{\mu\nu}$, and $\hat{R}^{\mu\nu}$; for example, equations (5.44) and (5.45) now take the form

$$T\{\hat{R}^{\mu\nu}-\tfrac{1}{2}\hat{G}^{\mu\nu}\hat{R}\}_{;\mu}=0,\qquad \hat{\Gamma}^\lambda\equiv T\{\hat{G}^{\mu\nu}\hat{\Gamma}^\lambda_{\mu\nu}\}=0$$

In conclusion, notice that Horowitz (1980, 1981) used an axiomatic approach to the construction of an expectation value of the stress tensor and obtained a nonlocal expression, which he then used as the right-hand side of the Einstein equation (6.104). More recently, this formalism was developed by Jordan (1986, 1987) (see also Biernacki and Królak, 1986).

7. PHYSICAL CONSEQUENCES OF THE THEORY WITH STOCHASTIC AND QUANTUM SPACE-TIME STRUCTURES

As shown above, the introduction of the hypothesis of stochastic and quantum fluctuating metrics into the physical theory leads to a change of the particle mass [(1.76), (1.79)] and the Newtonian potential (6.67) and also to the appearance of an additional force [(1.50), (1.55)] in the microworld. In this section we will consider other consequences of the theory of interest.

7.1. Speculation about Force of Inertia

The origin of inertia presents one of the fundamental problems of physical theory. Newton and Mach considered this problem in different ways. Newton assumed that inertial forces such as centrifugal ones must appear due to acceleration with respect to "absolute space," while Mach suggested that inertial forces are more probably generated by the general mass of heavenly bodies. The difference in their assertions is not metaphysical but physical, since if Mach were right, then a large mass would give rise to small alterations of the inertial forces near it, while if Newton were right, then such effects would not appear [for details and further discussion see Weinberg (1972) and Bertotti *et al.* (1984)]. Here our goal is modest; we consider only some possible explanations of the origin of inertial force from the point of view of stochastic and quantum fluctuations of the space-time metric.

We assume that, due to the existence of the cosmic background radiation stochastic field $\varepsilon_{\mu\nu}(x)$ [or quantized field $\hat{\varepsilon}_{\mu\nu}(x)$], the inertial system of reference is slightly changed [a useful discussion of such possibilities was presented by Bertotti *et al.* (1984)], in which there always appears some *additional* small "*potential*" *force*

$$\mathbf{F}^{(s,q)} = -\nabla \varphi_f = \nabla \phi_{(s,q)} \tag{7.1}$$

(Of course, the term "force" used here is not in the direct Newtonian sense and in general relativity it should be related to the gauge group concept, as in all contemporary geometrized versions of Maxwell and Yang-Mills gauge theories.) Where

$$\phi_s = \tfrac{1}{4}\varepsilon_{00}^2(x) + \tfrac{1}{8}\varepsilon_0^\rho(x)\varepsilon_{0\rho}(x)$$

and

$$\phi_q = \tfrac{1}{4}T[\hat{\varepsilon}_{00}^2(x) + \tfrac{1}{2}\hat{\varepsilon}_0^\rho(x)\hat{\varepsilon}_{0\rho}(x)]$$

in accordance with the existence of stochastic $\varepsilon_{\mu\nu}(x)$ and quantum $\hat{\varepsilon}_{\mu\nu}(x)$ fields, respectively. In the presence of a particle, it is suggested that both the distribution function $\tilde{D}(\mathbf{p}^2)$ in the correlation $\langle \varepsilon_{\mu\nu}(x)\varepsilon_{\rho\delta}(y)\rangle_\varepsilon$ and the form factor $V(\mathbf{p}^2)/\mathbf{p}^2$ in the causal Green function $\langle 0|T[\hat{\varepsilon}_{\mu\nu}(x)\hat{\varepsilon}_{\rho\sigma}(y)]|0\rangle$ depend on the momentum variable $\mathbf{p}_0$ of the particle and are located at the point $\mathbf{p}=\mathbf{p}_0$. With this assumption each particle, due to the stochastic (or quantized) field $\varepsilon_{\mu\nu}(x)$, generates around itself some "potential" force proportional to its momentum $\mathbf{p}_0$. For this purpose, let us consider the particular case when

$$V(\mathbf{p}^2)\mathbf{p}^{-2}=(\mathbf{p}-\mathbf{p}_0)^{-2}[1+(\Delta\mathbf{p}_0)^{-2}(\mathbf{p}-\mathbf{p}_0)^2]^{-2} \tag{7.2}$$

where

$$\Delta\mathbf{p}_0=m\,\Delta\mathbf{v}/\hbar; \qquad \mathbf{p}_0=m\mathbf{v}/\hbar$$

Then the "potential" force (7.1) when averaged takes the form

$$\begin{aligned}\mathbf{F}_q&=\mp\tfrac{1}{6}G(2\pi)^{-3}\int d^3p\;\mathbf{p}(\mathbf{p}-\mathbf{p}_0)^{-2}(\Delta\mathbf{p}_0)^4[(\Delta\mathbf{p}_0)^2+(\mathbf{p}-\mathbf{p}_0)^2]^{-2}\\&=\mp G\mathbf{p}_0(\Delta p_0)/48\pi\end{aligned} \tag{7.3}$$

where the sign $\mp$ depends on the definition of the factor $\exp(\mp i\mathbf{qx})$. This "force" has dimension of $[G/L^2]$, where

$$G=6.67\times10^{-11}\ \mathrm{N\,m^2/kg^2}, \qquad [p_0]=[L^{-1}]$$

Thus, each particle undergoes a proper action with the "force"

$$\mathbf{F}=\mp G\mathbf{p}_0(\Delta p_0)m^2/48\pi, \qquad \mathbf{p}_0=m\mathbf{v}/\hbar$$

due to the stochastic (or quantized) background radiation field $\hat{\varepsilon}_{\mu\nu}(x)$. When the particle is at rest, the force disappears and as soon as the particle starts to move, the "force" simultaneously begins to act on it. It is easily seen that even in the *macroworld this force becomes appreciable.* For a macroparticle it is reasonable to assume

$$|\mathbf{F}|\sim m^2G/48\pi(\Delta s)^2, \qquad \Delta s=v\,\Delta t$$

where Δt is a characteristic time during which the particle's velocity is noticeably changed. For example, let a particle with mass $m=10$ kg move with the change of velocity from zero to 1 m/sec during the time interval $\Delta t\sim10^{-6}$ sec; then the expected inertial force becomes sufficiently large:

$$|\mathbf{F}|=44.3\ \mathrm{N}$$

Notice that instead of (7.2), another form factor of the type

$$V_1(\mathbf{p}^2)/\mathbf{p}^2=\mathbf{p}^{-2}[1+(\Delta\mathbf{p}_0)^{-2}(\mathbf{p}-\mathbf{p}_0)^2]^{-n}, \qquad n\geq2$$

may be chosen, which leads to a complication in the calculation procedure, but the result remains the same.

7.2. Acceleration Mechanism of Cosmic Rays

Let us consider now other interesting problems, connected with the origin of cosmic rays in high-energy astrophysics. The acceleration mechanism that carries cosmic rays (particularly protons) to energies of 10^{20}–10^{22} eV in the primary cosmic radiation remains unsolved (Ginzburg and Ptuskin, 1976; Hillas, 1975). Namsrai (1986*b*) attempted to understand this problem from the point of view of the hypothesis of space-time stochasticity and fluctuations in the metric.

We show that this problem may be easily solved by using the inertial force (7.1). It is quite possible that cosmic-ray particles satisfy the *equation of motion*

$$m\frac{d\mathbf{v}}{dt} = \text{const}\cdot\mathbf{p} \tag{7.4}$$

in accordance with the definition of inertial force (7.1) and (7.3). This equation may be rewritten in the form

$$\frac{dE}{dt} = \text{const}\cdot E$$

with the solution

$$E = E_0 \exp(\text{const}\cdot t) \tag{7.5}$$

By appropriate choice of the constant in (7.5), it may be shown that during the time of evolution of the universe the cosmic-ray particle energy (mainly proton) reaches 10^{19}–10^{20} eV [for details, see Sinha and Roy (1986) and Namsrai (1986*b*)].

7.3. Quantum Mechanical Consideration

It turns out that our hypothesis of stochastic or quantum fluctuation in the metric leads to some interesting consequences due to the fact that the Hamiltonian of a physical system is changed in accordance with the formulas (1.75) and (1.78). We consider here the latter case only. Thus, for the form (1.18b) of distribution, the *Hamiltonian* of the system undergoes the following change:

$$\mathbf{p}^2/2m \Rightarrow (\mathbf{p}^2/2m)(1-\tfrac{5}{12}Gl^{-2}) \tag{7.6}$$

where we have taken into account the quantity

$$\tilde{D}_l(0) = G/l^2$$

for any distribution satisfying the condition

$$(2\pi)^{-4}\int d^4q\, D_l(q^2)=1$$

With the change (7.6), the *Schrödinger equation* acquires the form

$$\Delta\psi+(2m/\hbar^2)[E-U(r)-\hat{V}']\psi=0 \tag{7.7}$$

for the stationary states, where $U(r)$ is the external potential field and

$$\hat{V}'=\frac{\hbar^2}{2m}\frac{5}{12}Gl^{-2}\Delta \tag{7.8}$$

Here

$$\Delta=r^{-2}\,\partial(r^2\partial/\partial r)/\partial r-r^{-2}\hat{\mathbf{L}}^2$$

is the *Laplace operator* written in the spherical system of coordinates and $\hat{\mathbf{L}}$ is the angular momentum operator.

As usual, the solution of equation (7.7) is presented in the form

$$\psi=R(r)\,Y_{Lm}(\theta,\varphi)$$

where the angular part $Y_{Lm}(\theta,\varphi)$ satisfies the equation

$$\hat{\mathbf{L}}^2\,Y_{Lm}=L(L+1)\,Y_{Lm}$$

whence its radial part $R(r)=\chi(r)/r$ is defined by the equation

$$d^2\chi/dr^2+[(2m/\hbar^2)(1-\tfrac{5}{12}Gl^{-2})^{-1}(E-U)-L(L+1)r^{-2}]\chi=0$$

Due to the second term in (7.6) the energy level of an electron in an atom undergoes an additional shift given by the formula

$$\Delta E_n=\int d\mathbf{r}\,\psi_n^{(0)*}\hat{V}'\psi_n^{(0)} \tag{7.9}$$

Here we have assumed that in the given case the perturbation theory is applied, where $\hat{V}'$ represents a small correction ("disturbance") to the "unperturbed" operator $\hat{H}_0$.

It is interesting to calculate the correction to the *Lamb shift* $\Delta E(2S_{1/2}-2P_{1/2})$ due to a stochastic (or quantized) fluctuation in the metric. To define this correction, we consider the unperturbed normalized *wave function* written in atomic units

$$R_{20}=(1/\sqrt{2})\,e^{-r/2}(1-r/2),\qquad R_{21}=(1/2\sqrt{6})r\,e^{-r/2}$$

for the hydrogen atom ($Z=1$). Then taking into account (7.8) and (7.9), we have

$$\Delta E_{2s}^{(1)} = \tfrac{5}{48} G l^{-2} \int_0^\infty dr\, r^2\, e^{-r/2}(1-r/2)[d^2/dr^2 + (2/r)\, d/dr] \times e^{-r/2}(1-r/2) = -\tfrac{5}{96} G l^{-2}$$

and

$$\Delta E_{2p}^{(1)} = 5(24)^{-2} G l^{-2} \int_0^\infty dr\, r^2\, e^{-r/2}\, r \times [d^2/dr^2 + (2/r)\, d/dr - 2r^{-2}]\, e^{-r/2} r = -\tfrac{5}{144} G l^{-2}$$

Now the shift $\Delta E(2S_{1/2} - 2P_{1/2})$ in hydrogen takes the form, in natural units,

$$\Delta E(2S_{1/2} - 2P_{1/2}) = -\tfrac{5}{144} G l^{-2} m\alpha^2 \hbar^{-2}/2 = -\tfrac{5}{144} G l^{-2} \cdot Ry \tag{7.10}$$

where $\mathrm{Ry} = m\alpha^2/2\hbar^2$ is the *Rydberg constant*. The agreement between ΔE_{cal} and ΔE_{exp} in quantum electrodynamics within the present accuracy is $\sim 10^{-11}$ Ry (Erickson and Yennie, 1965; Lundeen and Pipkin, 1981); from this we obtain the following estimation:

$$l \gtrsim 10^{-28}\ \mathrm{cm}$$

7.4. Relativity, Anisotropy of Inertia, and the Value of the Fundamental Length

Owing to the above considerations, the stochastic (or quantized) nature of the space-time metric at short distances, after averaging (or taking the vacuum expectation) over a large scale, plays a role in the formation of an anisotropy of the universe and, in turn, gives rise to a slight change of the laws of motion of a particle in the inertial system of reference. It is natural to assume that the appearance of anisotropy is caused by the additional force obtained in the previous sections. In other words, this force may be understood as the source of a small difference in the values of gravitational and inertial masses.

On the level of the usual theory of gravity, in connection with the verification of Mach's principle of the possible influence of large mass accumulation (for example, in the presence of the Milky Way) on the laws of motion of a particle, experiments (Hughes *et al.*, 1960; Drever, 1961) devoted to testing the existence of a small difference in inertial mass have been carried out. Hughes and his team observed resonance absorption of photons by ^{7}Li nuclei in a magnetic field. The experimental result is that if one can represent a nucleus of ^{7}Li as a proton with angular momentum $J=\frac{3}{2}$ which is connected with other nucleons in a central symmetric potential, then the anisotropy of the proton mass Δm must be equal to

$$\Delta E = \Delta(p^2/2m) \approx (\Delta m/m)(p^2/2m) \lesssim 5.3\times 10^{-21}\ \text{MeV} \qquad (7.11)$$

where $p^2/2m$ is the kinetic energy of the proton. Since $p^2/2m$ is larger than $\frac{1}{2}$ MeV, this is reduced to the assertion that the *anisotropy of inertial mass* is bounded by (Weinberg, 1972)

$$\Delta m/m \sim 10^{-20} \qquad (7.12)$$

We know that in space-time with a stochastic (or quantized) fluctuation in the metric the kinetic energy of the particle is changed in accordance with formulas (1.75) and (1.78). This in turn gives an additional energy shift (7.9) for atomic level in the stationary case. We assume that this change of energy level in ^{7}Li is connected with the anisotropy of the proton mass given by (7.11) or (7.12).

Thus, first we write the change of kinetic energy due to the stochastic (or quantized) nature of the space-time metric by means of the *anisotropy of inertial mass*

$$p^2/2m \Rightarrow \mathcal{P}^2/2m = p^2/2(m-\Delta m) = (p^2/2m)(1+\Delta m/m)$$

Second, this change is connected with the shift of the atomic energy level given by (7.9). Now let us calculate this shift for the case of $L=1$, $n=2$, $Z=6$. The wave function for the basic states, i.e., unperturbed energy level, is

$$\psi_{nL} = R_{nL} Y_{Lm}$$

where

$$R_{21} = (1/2\sqrt{6})\, e^{-r/2} r$$

for $Z=1$. For hydrogenlike atoms it takes the standard form

$$R_{nL} \Rightarrow f_{nL}(r) = N_{nL}(2Zr/n)^L F(-n+L+1,\ 2L+2,\ 2Zr/n)\, e^{-Zr/n} \qquad (7.13)$$

where

$$N_{nL} = [(2L+1)!]^{-1}\{[(n+L)!]/2n(n-L-1)!\}^{1/2}(2Z/n)^{3/2}$$

is the normalized coefficient (for details, see Landau and Lifschitz, 1963) and $F(a, c, z)$ is the *degenerate hypergeometric function*. For our case, the expression (7.13) becomes

$$f_{21} = (1/2\sqrt{6}) Z^{5/2} r\, e^{-Zr/2} \qquad (7.14)$$

With this radial function, the energy shift due to a stochastic (or quantized) metric is easily calculated; the result reads

$$\Delta E_{2P}^{(1)} = \tfrac{5}{576} G l^{-2} Z^5 \int_0^\infty dr\, r^3\, e^{-Zr/2}$$
$$\times [d^2/dr^2 + (2/r)\, d/dr - 2r^{-2}] r\, e^{-Zr/2}$$
$$= -\tfrac{5}{288} G l^{-2} Z^2 (1+Z) \tag{7.15}$$

for the distribution function $D_l(q^2)$ satisfying the condition (1.18b). The formula (7.15) is expressed in atomic units. Thus, assuming $Z = 6$, we get

$$\Delta E_{2P}^{(1)} = -\tfrac{35}{4} G l^{-2} \cdot \mathrm{Ry} \cdot (m_p/m_e) \tag{7.16}$$

where $hc \cdot \mathrm{Ry} = m_e e^4 hc/4\pi\hbar^3 = 13.6$ eV, and $a = 0.529 \times 10^{-8}$ cm is the *Bohr radius.* On the other hand, relation (7.16) is bounded by the experimental value (7.11). Therefore, one can obtain the following estimation on the lower value of the fundamental length:

$$l \gtrsim \times 10^{-23}\ \mathrm{cm} \tag{7.17}$$

Thus, we see that the anisotropy property of inertia is very sensitive to the quantum (or stochastic) fluctuation of the space-time metric at short distances. Of course, the latter gives rise to the appearance of the slight anisotropy of the universe. On the other hand, from results given in Namsrai (1986*a*) it follows that

$$l \lesssim 10^{-22}\ \mathrm{cm}$$

Therefore, the value of the *fundamental length* lies in the interval

$$10^{-23} \lesssim l \lesssim 10^{-22}\ \mathrm{cm} \tag{7.18}$$

This result is crucial in our scheme.

7.5. Derivation of Upper Bound on the Value of the Fundamental Length from High-Energy Physics and Its Possible Scales

It is well known that the high-energy colliding-beam experiments allow one to probe very short distances and in turn space-time structures. The main components of the colliding-beam experiments are storage rings in which high-energy particles are accelerated to expected limiting energies. Among them, e^+e^- and $p\tilde{p}$ beams are crucial for space-time and matter structure investigations. Major experiments in these directions may be undertaken upon the completion of the LEP electron–positron accelerated machine with collison energies of around 100 GeV at CERN, the Tevatron collider complex at Fermilab, and others still under construction, including the proposed U.S. Superconducting Supercollider (SSC), an 85-m, ring to provide 20,000-GeV (20-TeV) proton beams, and the 3-TeV superconducting accelerator and storage complex (UNK) at Protvino, Moscow Region, USSR.

In this section we attempt to obtain the upper bound on the value of the fundamental length from high-energy experimental data, and speculate upon the possible existence of the energy scales $E_{\rm EW} \sim 100$ GeV and $E_{\rm NW} \sim 5$ TeV of the unification of electromagnetic and weak, and weak and nuclear processes, respectively. The former is called the electroweak unification (interaction), following S. Weinberg, A. Salam, and S. L. Glashow. The latter possibility is very interesting for the experimental verification of the theory.

In order to deal with the theory of fundamental length, we consider the simple form of the *Lagrangian function* (Namsrai, 1985)

$$\mathscr{L} = \tfrac{1}{2}\varphi_R(x)\hat{\mathscr{P}}^2_{\varphi_R}(x)$$

where $\varphi_R(x)$ is a massless scalar field and

$$\hat{\mathscr{P}}^2 = \Box \cosh^2(l\sqrt{-\Box})$$

This type of momentum operator $\hat{\mathscr{P}}$ was also discussed by Fujiwara (1980) from the point of view of three-dimensional quantized space. The *equation of motion* of this field is obtained in the usual manner (the principle of stationary action)

$$\Box \cosh^2(l\sqrt{-\Box})\varphi_R(x) = 0 \tag{7.19a}$$

or in the massive particle case

$$(\Box - m^2)\cosh^2[l(-\Box + m^2)^{1/2}]\varphi_R(x) = 0 \tag{7.19b}$$

For the photon field $A_\mu(x)$ without the source field $J_\mu(x)$,

$$\mathscr{L}_{\rm em} = -\tfrac{1}{2}[\hat{\mathscr{P}}_\nu A_\mu(x)]^2 \tag{7.20a}$$

and

$$\Box \cosh^2[l\sqrt{-\Box}]A^R_\mu(x) = 0 \tag{7.20b}$$

The applicability of the choice of Lagrangian form for the electromagnetic field has been discussed by 't Hooft and Veltman (1973).

We see that these equations are differential equations of infinite order, i.e., they are in fact integral equations. In order to solve the Cauchy problem, we have to know the values of the functions $\varphi_R(x)$ and all its derivatives at the initial moment of time. Thus, unlike the usual fields obeying differential equations of finite order (in most cases, second order), we obtain new objects—*nonlocal* (*extended*) *fields* of the Efimov (1977) type. We denote these objects by index R; for example, $\varphi_R(x)$ and $A_\mu^R(x)$ are extended scalar and photonlike fields, respectively. Here, we are interested only in the Green functions of equations (7.19b) and (7.20b).

Thus, our formalism coincides with the usual scheme of quantum field theory in the free particle case and therefore gives no new information in this case. However, in the virtual states of the particles, the formalisms are essentially different. We now consider this situation. The main object of the virtual state of a particle is its Green's function (or propagator). As is well known (see, for example, 't Hooft and Veltman, 1973), the propagators are minus the inverse of the operator found in the quadratic term of the *free Lagrangian*, for example,

$$\mathscr{L} = \tfrac{1}{2}\varphi(x)(\Box - m^2)\varphi(x) \Rightarrow (m^2 + k^2 - i\varepsilon)^{-1}$$

This rule reads for the Lagrangian (7.20a)

$$\tilde{D}^R_{\mu\nu}(k) = g_{\mu\nu}/[(k^2 - i\varepsilon)\cosh^2(l\sqrt{k^2})] \tag{7.21}$$

On the other hand, the *Green function* $D^R_{\mu\nu}(x)$ is the solution of the equation

$$\Box\cosh^2(l\sqrt{-\Box})D^R_{\mu\nu}(x) = g_{\mu\nu}\delta^{(4)}(x) \tag{7.22}$$

The solution [the causal Green function $D^R_{\mu\nu}(x)$] to this equation is given by the contour integral:

$$\begin{aligned} D^R_{\mu\nu}(x) &= -g_{\mu\nu}D^R(x) \\ &= -g_{\mu\nu}i^{-1}(2\pi)^{-4}\int_c d^4k\, e^{-ikx}(k^2 - i\varepsilon)^{-1}\cosh^{-2}(l\sqrt{k^2}) \end{aligned} \tag{7.23}$$

The contour of integration c is chosen as in the usual local theory and is determined by the "$i\varepsilon$ rule".

It is important to notice that in our scheme *ultraviolet divergences* are absent, since $D^R_{\mu\nu}(0) < \infty$; for example,

$$D^R(0) = -i^{-1}(2\pi)^{-4}\int_c d^4k\,(k^2 - i\varepsilon)^{-1}\cosh^{-2}(l\sqrt{k^2}) < \infty$$

Indeed, after transformation to the Euclidean metric, we get

$$\begin{aligned} D^R(0) &= -\pi^2(2\pi)^{-4}\int_0^\infty du\,\cosh^{-2}(l\sqrt{u}) \\ &= -2\pi^2(2\pi)^{-4}\int_0^\infty dx\,x\cosh^{-2}(lx) \\ &= -\tfrac{1}{8}\ln 2\cdot\pi^{-2}l^{-2} \end{aligned}$$

At the same time as the photon propagator (7.23), the *Coulomb law* is also changed. Thus, the potential of two interacting charged particles acquires the following form in the static limit:

$$U_C(r) = e^2(2\pi)^{-3} \int d^3p\, \mathbf{p}^{-2}\, e^{-i\mathbf{pr}} \cosh^{-2}(l\sqrt{\mathbf{p}^2})$$

$$= (e^2/2\pi^2 r) \int_0^\infty dx\, x^{-1} \sin(xr) \cosh^{-2}(lx) \qquad (7.24)$$

From this, it is easy to see that $U_C(0) < \infty$; indeed,

$$U_C(0) = (e^2/2\pi^2) \int_0^\infty dx \cosh^{-2}(lx) = (-1/2\pi^2 l)e^2$$

We now give the *Mellin representation* for the propagator $\tilde{D}^R_{\mu\nu}(k)$ of the photon field. For this, making use of the expansion for $\cosh^{-2} x$,

$$\cosh^{-2} x = 4/(e^{2x} + 2 + e^{-2x}) = -4 \sum_{n=1}^{\infty} (-1)^n n\, e^{-2nx} \qquad (7.25)$$

we get

$$\cosh^{-2} x = 4 \sum_{n=1}^{\infty} (-1)^{n+1} n \sum_{k=0}^{\infty} (-1)^k \frac{(2nx)^k}{k!}$$

$$= 4 \sum_{n=1}^{\infty} (-1)^{n+1} n \frac{1}{2i} \int_{-\beta+i\infty}^{-\beta-i\infty} \frac{d\rho}{\sin \pi\rho} \frac{(2nx)^\rho}{\Gamma(1+\rho)} \qquad (7.26)$$

where $1 > \mathrm{Re}\,\beta > 0$. Using the properties of the $\Gamma(x)$ function, it is possible to move the contour of integration in expression (7.26) to the left through the point $\rho = -1$, and in the obtained results one can take

$$\sum_{n=1}^{\infty} (-1)^{n+1} n^{1+\rho} = (1 - 2^{2+\rho})\zeta(-1-\rho)$$

since $\mathrm{Re}(-1-\rho) > 0$, where $\zeta(z)$ is the *Riemann zeta function* having a single pole at the point $z = 1$ and satisfying the following conditions:

$$2^{1-z}\Gamma(z)\zeta(z) \cos \tfrac{1}{2}\pi z = \pi^z \zeta(1-z)$$

$$\zeta(2m) = 2^{2m-1}\pi^{2m}|B_{2m}|/(2m)!$$

$$\zeta(-2m) = 0$$

$$\zeta(1-2m) = -B_{2m}/2m, \qquad m = 1, 2, 3, \ldots$$

In particular,

$$\zeta(0)=\tfrac{1}{2}, \qquad \zeta(-1)=-B_2/2=-\tfrac{1}{12}$$

Here B_m are the *Bernoulli numbers*; for example,

$$B_0=1, \qquad B_1=-\tfrac{1}{2}, \qquad B_2=\tfrac{1}{6}$$

As a result, we obtain

$$\cosh^{-2} x=\frac{1}{2i}\int_{-\gamma+i\infty}^{-\gamma-i\infty} d\rho \frac{v(\rho)}{\sin \pi\rho} \frac{x^\rho}{\Gamma(1+\rho)}, \qquad 2<\gamma<1 \tag{7.27}$$

where

$$v(\rho)=4\cdot 2^\rho(1-2^{2+\rho})\zeta(-1-\rho) \tag{7.28}$$

In particular,

$$v(-1)=-1, \qquad v(0)=0, \qquad v(1)=0, \qquad v(2)=2$$

After these simple calculations we have the following *Mellin representation* for the photon propagator:

$$\tilde{D}^R_{\mu\nu}(k)=-g_{\mu\nu}\frac{1}{2i}\int_{-\gamma+i\infty}^{-\gamma-i\infty} d\rho \frac{v(\rho)}{\sin \pi\rho\cdot\Gamma(1+\rho)} l^\rho(k^2-i\varepsilon)^{\rho/2-1}, \qquad 2<\gamma<1 \tag{7.29}$$

Representations (7.27) and (7.29) are very convenient for the purpose of concrete calculation. For example, by using the representation (7.27), the expression (7.24) for the potential $U_C(r)$ is calculated explicitly and takes the form

$$\begin{aligned} U_C(r)&=\frac{1}{2\pi^2 r}\frac{1}{2i}\int_{-\gamma+i\infty}^{-\gamma-i\infty} d\rho \frac{v(\rho)}{\sin \pi\rho\cdot\Gamma(1+\rho)} \\ &\quad\times \sin\tfrac{1}{2}\pi\rho\cdot\Gamma(\rho)\left(\frac{l}{r}\right)^\rho \\ &=\frac{1}{2\pi^2 r}\frac{1}{2i}\int_{-\gamma+i\infty}^{-\gamma-i\infty} d\rho \frac{v(\rho)}{2\rho\cos\tfrac{1}{2}\pi\rho}\left(\frac{l}{r}\right)^\rho, \qquad 1<\gamma<0 \end{aligned} \tag{7.30}$$

Here the following integral is used:

$$\int_0^\infty dx\, x^{\rho-1}\sin ax=a^{-\rho}\Gamma(\rho)\sin\tfrac{1}{2}\pi\rho, \qquad a>0, \quad 0<|\text{Re}\,\rho|<1$$

Two cases should be distinguished: $l/r>1$ (i.e., $r\to 0$) and $l/r<1$ ($r\to\infty$). In both the first and second cases it is necessary to move the contour of integration in (7.30) to the left and to the right, respectively. Thus, we have

$$\begin{aligned} U_C(r) &= -(e^2/2\pi^2 r)[-v(-1)(l/r)^{-1}+\tfrac{1}{3}v(-3)(l/r)^{-3}+I'(r/l)] \\ &= -(e^2/2\pi^2 l)-e^2r^2l^{-3}/144+I(r/l) \\ I(r/l) &= 7e^2\pi^2r^4l^{-5}/[(24)^2 100] \end{aligned} \tag{7.31}$$

for $r<l$, and

$$U_C(r)=e^2/4\pi r$$

for $l<r$. The function $U_C(r)$ at $r=0$ represents the so-called proper electrostatic energy of the electron in the classical field theory. As seen above, in our model the proper energy of the electron is finite, $U_C(0)\sim e^2/l$. This result coincides with the well-known classical electrodynamic value $U_C(0)\sim e^2/a$, where a is the electron size (classical). In the last case it is usually assumed that the electron is a pointlike object with radius a. However, in our case there is an interesting possibility: due to the minus sign of $U_C(r)$ for $r=0$ (see Figure 1a) two electrons may form a whole bounded state, i.e., unlike the usual classical theory, in quantized space-time the electrical repulsion between two electrons becomes an electrical attraction at small distances.

On the other hand, at distances $r>l$ our potential $U_C(r)$ reproduces exactly the Coulomb law [without any terms of the type $e^2(l^2/r^3),\ldots$]. This means that quantum electrodynamics is a *beautiful local theory* up to distances l; if the true length is $\sim 10^{-33}$ cm, then QED becomes local once and for all.

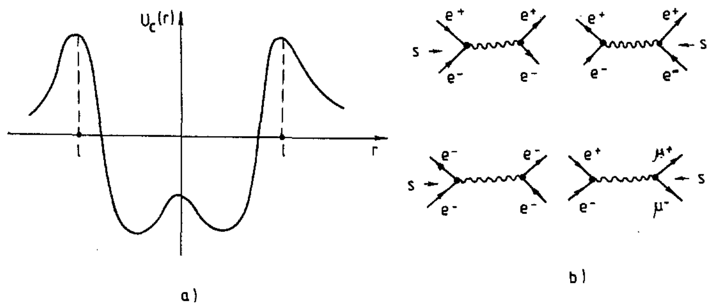

Fig. 1. (a) Illustration of the change of the Coulomb potential due to quantized space-time at short distances. (b) Diagrams of electromagnetic leptonic processes giving the main contribution to the scattering value at high energies.

An analogous calculation for equation (7.19b) gives in the Euclidean metric

$$\tilde{D}^c_R(p) = (m^2+p^2)^{-1} \cosh^{-2}[l(m^2+p^2)^{1/2}] \tag{7.32}$$

and therefore the *causal Green function* of a scalar massive particle

$$D^c_R(x) = (2\pi)^{-4} \int d^4p\, e^{-ipx} \tilde{D}^c_R(p) \tag{7.33}$$

is finite at the point $x=0$. Indeed, the function $D^c_R(0)$ in the Euclidean metric is given by

$$\begin{aligned} D^c_R(0) &= \pi^2(2\pi)^{-4} \int_0^\infty du\, u(m^2+u)^{-1} \cosh^{-2}[l(m^2+u)^{1/2}] \\ &= \frac{1}{16\pi^2} \frac{1}{2i} \int_{-\gamma+i\infty}^{-\gamma-i\infty} d\rho \frac{v(\rho)}{\sin \pi\rho} \frac{m^{2+\rho}}{\Gamma(1+\rho)} l^\rho \frac{\Gamma(-1-\frac{1}{2}\rho)}{\Gamma(1-\frac{1}{2}\rho)} \\ &= \frac{m^2}{4\pi^2} \frac{1}{2i} \int_{-\gamma+i\infty}^{-\gamma-i\infty} d\rho \frac{v(\rho)}{\sin \pi\rho} \frac{(ml)^\rho}{\Gamma(1+\rho)} \rho^{-1}(2+\rho)^{-1} \\ &= \frac{1}{8} \frac{m^2}{\pi^2} [\ln 2 \cdot m^{-2} l^{-2} + v'(0) + \ln ml - \Psi(1) - \tfrac{1}{2}] 3 < \gamma < 2, \quad \Psi(1) = -C \end{aligned}$$

where $C \simeq 0.577$ is the Euler constant.

Similarly to formula (7.29), the following *Mellin representation* holds for the function (7.32):

$$\tilde{D}^c_R(p) = \frac{1}{2i} \int_{-\gamma+i\infty}^{-\gamma-i\infty} d\rho \frac{v(\rho)}{\sin \pi\rho \cdot \Gamma(1+\rho)} l^\rho (m^2+p^2-i\varepsilon)^{\rho/2-1}, \qquad 2<\gamma<1 \tag{7.34}$$

Here $v(\rho)$ is given by formula (7.28).

As in the case of the Coulomb law, in the given scheme the *Yukawa potential* between two scalar particles acquires the form

$$\begin{aligned} U_Y(r) &= g^2(2\pi)^{-3} \int d^3p\, (m^2+\mathbf{p}^2)^{-1} e^{-i\mathbf{pr}} \cosh^{-2}[l(m^2+\mathbf{p}^2)^{1/2}] \\ &= \frac{g^2}{2\pi^2\sqrt{\pi} r} \frac{m}{2i} \int_{-\gamma+i\infty}^{-\gamma-i\infty} d\rho \frac{v(\rho) l^\rho}{\sin \pi\rho \cdot \Gamma(1+\rho)} \left(\frac{2m}{r}\right)^{(\rho-1)/2} \\ &\quad \times \cos \tfrac{1}{2}\pi(\rho-1)\Gamma(\tfrac{1}{2}\rho) K_{(\rho+1)/2}(mr), \qquad 4<\gamma<3 \end{aligned} \tag{7.35}$$

where $K_\nu(x)$ is the MacDonald function of νth order. This representation is valid for the case $r=0$ and has only a single pole at the point $\rho=0$. The calculation of the *residue* at this point gives

$$U_Y(r)=(g^2/4\pi r)\,e^{-mr} \qquad \text{for} \quad r\neq 0 \tag{7.36}$$

In order to calculate the value $U_Y(r)$ for $r\to 0$, we use another representation obtained directly from the first equality in (7.35):

$$\lim_{r\to 0} U_Y(r)=U_Y(0)+r^2U_Y^1+O(r^4)$$

where

$$\begin{aligned} U_Y(0)&=\frac{g^2}{2\pi^2}\int_0^\infty dx\,x^2(m^2+x^2)^{-1}\cosh^{-2}[l(m^2+x^2)^{1/2}] \\ &=\frac{mg^2}{4\pi^2}\frac{1}{2i}\int_{-\gamma+i\infty}^{-\gamma-i\infty} d\rho\,\frac{v(\rho)}{\sin\pi\rho}\frac{(ml)^\rho}{\Gamma(1+\rho)}\frac{\Gamma(\frac{3}{2})\Gamma(-\frac{1}{2}-\frac{1}{2}\rho)}{\Gamma(1-\frac{1}{2}\rho)} \\ &=-\frac{g^2}{2\pi^2 l}-\frac{mg^2}{4\pi}-\frac{7}{4}m\frac{g^2}{\pi^4}(ml)\zeta(3)+O(m^2l^2) \end{aligned} \tag{7.37a}$$

$$\begin{aligned} U_Y^1(0)&=-\frac{g^2}{2\pi^2}\frac{1}{3!}\int_0^\infty dx\,x^4(m^2+x^2)^{-1}\cosh^{-2}[l(m^2+x^2)^{1/2}] \\ &=-\frac{g^2}{2\pi^2}\frac{1}{3!}\frac{1}{2}m^3\frac{1}{2i}\int_{-\gamma+i\infty}^{-\gamma-i\infty} d\rho\,\frac{v(\rho)(ml)^\rho}{\sin\pi\rho\cdot\Gamma(1+\rho)}\frac{\Gamma(\frac{5}{2})\Gamma(-\frac{3}{2}-\frac{1}{2}\rho)}{\Gamma(1-\frac{1}{2}\rho)} \\ &=-\frac{g^2}{144l^3}-\frac{g^2m^2}{8\pi^2 l}-\frac{g^2m^3}{24\pi}+O(ml) \end{aligned} \tag{7.37b}$$

Here $\zeta(3)=\sum_{n=1}^\infty (1/n^3)=1.20205690$.

Combining the formulas (7.36), (7.37a), and (7.37b), we have

$$U_Y(r)=\begin{cases} -g^2/2\pi^2 l-(m/4\pi)g^2-\frac{7}{4}mg^2(ml/\pi^4)\zeta(3)+O(m^3l^3) \\ \quad -r^2(g^2/144l^3+N) \qquad \text{for} \quad r\to 0 \\ (g^2/4\pi r)\,e^{-mr} \qquad \text{for} \quad r\neq 0 \end{cases} \tag{7.38}$$

where

$$N=(g^2/8\pi^2)(m^2/l)+(g^2/24\pi)m^3+O(ml)$$

Thus, we see that the *Yukawa law* is valid up to the point $r=0$, and therefore, the corresponding theory is local almost everywhere.

Now we discuss equation (7.19b). It has three solutions:

$$\begin{aligned}
&(1)\quad p^2=-m^2, \text{ at which } \cosh 0=1\\
&(2)\quad p^2\neq -m^2,\quad \cos[l(-p^2-m^2)^{1/2}]=0,\quad p^2<-m^2 \qquad (7.39)\\
&(3)\quad \text{Trivial solution } \varphi_R(p)\equiv 0,\qquad \text{for } -m^2<p^2
\end{aligned}$$

The first corresponds to the free scalar particle with mass m and second to the family of particles with masses

$$M_n=\{m^2+[(\pi/l)(\tfrac{1}{2}+n)]^2\}^{1/2},\qquad n=0,1,2,\ldots \qquad (7.40)$$

In the second case the initial particle becomes a virtual one, but at the same time a family of particles is generated due to the quantized space-time properties at short distances. On the other hand, these new generated particles may be understood as excited states of the initial particle with discrete energy levels

$$E_n=(E_0^2+E_n'^2)^{1/2} \qquad (7.41)$$

in quantized space-time, where

$$E_0=(m^2+\mathbf{p}^2)^{1/2},\qquad E_n'=(\pi/l)(\tfrac{1}{2}+n)$$

It is not difficult to construct finite quantum electrodynamics (Namsrai, 1985) with the propagator (7.21) in accordance with the prescription developed by Efimov (1977, 1985) and Namsrai (1986*b*). Here we obtain only an upper bound on the value of the fundamental length from experimental data on high-energy scattering processes. Since electromagnetic processes of the type $e^-e^-\to e^-e^-$, $e^+e^-\to e^+e^-$, and $e^+e^-\to\mu^+\mu^-$ are described even by a low order of the perturbation theory up to, at high energies, the recently attainable one (see Figure 1b), the ratio of cross sections calculated by the usual local and nonlocal theories discussed above is given by

$$\sigma_{\text{nonloc}}/\sigma_{\text{loc}}=[V(-sl^2)]^2=[1-\tfrac{1}{2}v(2)sl^2]^2\approx 1-2sl^2$$

where $v(2)$ is given by the expression (7.28), and

$$s=(p_1+p_2)^2=(2E)^2=W^2$$

$W=2E$ is total energy in the center-of-mass frame of reference.

An estimation based on this formula is very simple, and using present experimental data (see, for example, Bartel *et al.*, 1980; Berger *et al.*, 1980) we have

$$l\lesssim 10^{-16}\text{ cm} \qquad (7.42)$$

Of course, there exist other bounds on the fundamental length [see, for example, Bracci *et al.* (1983) and previous sections].

Now we determine the *energy scales* at which different forces are unified. This is also possible in our model. For this purpose, first we construct the weak potential between two particles. In accordance with the traditional method, we do this in the following manner:

$$\tilde{D}^c_W(p) = (g_{\mu\nu} - p_\mu p_\nu / m_W^2)(m_W^2 + p^2 - i\varepsilon)^{-1} \underset{m_W \to \infty}{\Rightarrow} g_{\mu\nu}/m_W^2$$

or in the language of Feynman diagrams, the intermediate vector weak interaction transforms into the four-fermion weak interaction, i.e., two vertices of interacting fermions linked with the propagator of the intermediate boson at the limit $m_W \to \infty$ become one vertex with four fermion lines entering into it. Upon this, the *weak potential* acquires the form

$$U_W^{\rm loc}(r) = g_W^2 (2\pi)^{-3} \int d^3p\, e^{-i\mathbf{pr}} (m_W^2 + \mathbf{p}^2)^{-1} (\delta_{ij} - p_i p_j / m_W^2)$$

$$\Rightarrow (G_F/\sqrt{2})(2\pi)^{-3} \delta_{ij} \int d^3p\, e^{-i\mathbf{pr}}$$

where $G_F/\sqrt{2} = g_W^2/m_W^2$. It is a local case. In our model it takes the form

$$U_W(r) = (G_F/\sqrt{2})(2\pi)^{-3} \int d^3p\, e^{-i\mathbf{pr}} \cosh^{-2}(l\sqrt{\mathbf{p}^2})$$

$$= (G_F/\sqrt{2})(1/2\pi^2 l^2 r) \int_0^\infty dy\, y \sin(ry/l) \cosh^{-2} y$$

$$= -(G_F/\sqrt{2})(1/2\pi^2 l^2 r)\, d(\pi z / 2 \sinh \tfrac{1}{2}\pi z)/dz$$

$$= -(G_F/\sqrt{2})(1/4\pi l^2 r)$$

$$\times [1/\sinh \tfrac{1}{2}\pi z - \tfrac{1}{2}\pi z \cosh(\tfrac{1}{2}\pi z)/\sinh^2 \tfrac{1}{2}\pi z]$$

where $z = r/l$. It is easy to verify that this potential is finite at the point $r = 0$. For $r \to 0$, we have

$$U_W(r) = (G_F/\sqrt{2})(1/24 l^3) - (G_F/\sqrt{2})(\pi^2/2^5 l^5)(31/180) r^2 \quad (7.43)$$

We now assume that the electrostatic energy $U_C(0)$ and the weak-static energy $U_W(0)$ of the electron coincide with the absolute value at the energy scale given by $E_{ew} = \hbar / l_{ew} c$. Here we call E_{ew} the *electroweak energy scale* at which electromagnetic and weak interactions are unified. Thus, from

(7.31) and (7.43), we have

$$(e^2/2\pi^2 l_{ew}) = (G_F/\sqrt{2})(1/24 l_{ew}^3)$$

or

$$E_{ew} = (\alpha/G_F)^{1/2}(48\sqrt{2}/\pi)^{1/2} = 118.1 \text{ GeV}$$

where $\alpha = e^2/4\pi$ and $E_{ew} = \hbar/l_{ew}c$. We see that the obtained energy scale is closer to the unified scale of electroweak interactions due to S. Weinberg, A. Salam, and S. Glashow, i.e., it coincides roughly with the mass of the $W^{\pm}$ and Z^0 bosons.

Analogously, comparing the values of the weak-static energy $U_W(0)$ and strong Yukawa energy $U_Y(0)$ at the same energy scale $E_{nw} = \hbar/l_{nw}c$ (we call this the *nuclear-weak energy scale*), we have from (7.38) and (7.43):

$$(g^2/2\pi^2 l_{nw}) = (G_F/\sqrt{2})(1/24 l_{nw}^3)$$

or

$$E_{nw} = (f/G_F)^{1/2}(48\sqrt{2}/\pi)^{1/2} = 5353 \text{ GeV}$$

where

$$f = g^2/4\pi \sim 15, \qquad E_{nw} = \hbar/l_{nw}c$$

It is interesting to notice that the hypothesis of quantized space-time may indicate the energy scale of a grand unified theory linking weak, strong, and electromagnetic interactions at very high energy. It is no exception that this unification takes place at the energy scale $E_{nw} = 5353$ GeV (or, equivalently, at the distances $l \sim 4 \times 10^{-18}$ cm), which is much lower than the energy scale 10^{15} GeV discussed in the grand unified theory.

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